Lecture Notes
in Control and Information Sciences 272

Editors: M. Thoma · M. Morari

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Tao Yang

Impulsive Control Theory

With 29 Figures

Springer

Author

Dr. Tao Yang
University of California at Berkeley
Department of Electrical Engineering and Computer Sciences
Berkeley, CA 94720
USA

Cataloging-in-Publication Data applied for
Die Deutsche Bibliothek – CIP-Einheitsaufnahme
Yang, Tao:
Impulsive control theory / Tao Yang.
Berlin; Heidelberg; NewYork; Barcelona; Hong Kong; London; Milano; Paris; Tokyo:
Springer, 2001

(Lecture Notes in control and information sciences; 272)
ISBN 3-540-42296-X

ISBN 3-540-42296-X Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science + Business Media GmbH

http://www.springer.de

Typesetting: Digital data supplied by author. Data-conversion by PTP-Berlin, Stefan Sossna
Cover-Design: design & production GmbH, Heidelberg
Printed on acid-free paper SPIN 10843971 62/3020Rw - 5 4 3 2 1 0

This book is dedicated to my parents
and to
my wife

谨献给

敬爱的父母杨林保、邹恒清、爱妻杨春梅

Preface

The concept of impulsive control and its mathematical foundation called *impulsive differential equations*, or *differential equations with impulse effects*, or *differential equations with discontinuous righthand sides* have a long history. In fact, in mechanical systems impulsive phenomena had been studied for a long time under different names such as: mechanical systems with impacts. The study of impulsive control systems (control systems with impulse effects) has also a long history that can be traced back to the beginning of modern control theory. Many impulsive control methods were successfully developed under the framework of optimal control and were occasionally called *impulse control*. The so called impulse control is not exactly the impulsive control as will be defined in this book. The reader should not mix up these two kinds of control methods though in many papers they were treated as the same. Recently, there is a tendency of integrating impulsive control into *hybrid control systems*. However, this effort does not have much help to the development of impulsive control theory because impulsive systems can only be studied by the very mathematical tool based on impulsive differential equations. The effort to invent a very general framework of hybrid control system for studying impulsive control and other hybrid control problems will contribute no *essential* knowledge to impulsive control.

On the other hand, the long history of impulsive differential equations and impulsive control systems did not mean that we already had a good understanding of impulsive control systems. This is because for many years, the study of impulsive control problems had been restricted to only a few kinds of special problems such as mechanical systems with impacts and the optimal control of spacecraft. Another fact contributed to the slow development of impulsive control is that the early research activities were reported as Russian literature and therefore was not well-known to the English community. Only within the last two decades impulsive differential equations had been intensively studied in English literature and at least 7 English books had been published on impulsive differential equations.

Even after the publication of many English books on impulsive differential equations during the period of 1982 to 1995, the control community still saw nothing exciting about these mathematical tools because the well-known plants that can be studied by these mathematical tools seem to be too limited.

For example, mechanical systems with impacts are not a main focus of control community, predator-prey systems can not attract serious attention of control engineers. Unfortunately, mathematicians only know the above few kinds of real examples that fall into the scope of impulsive differential equations. And to make things worse, the existing monographs on impulsive differential equations target mainly mathematicians as potential readers.

Fortunately, this slow developing pace of impulsive control system had been changed at the end of last century because of the following facts:

- ♣ the theory of impulsive differential equations had been gradually diffused into control community;
- ♣ much more new plants, which can be modeled by impulsive differential equations, were found such as nanoelectronic devices and chaotic spread-spectrum communication systems . More important, these plants are of great interests to electrical engineers because of their industrial applications.

This is the first one of two books that are dedicated to impulsive systems and control. The second one entitled "Impulsive Systems and Control: Theory and Applications" will be published by *Nova Science Publishers, Inc.* [44]. In this book, the emphasis is put on the theoretical aspects of impulsive control systems. Therefore, the existence and stability of impulsive control strategies are studied in a very detailed manner. In the second book, the emphasis will be put on the applications of impulsive control theory. Both books will benefit three parties:

1. give mathematicians the real applications of impulsive differential equations and evoke new activities in pure and applied mathematical researches on impulsive differential equations;
2. provide engineers with a tool box and a well organized mathematical theory for impulsive control problems.
3. provide physicists and engineers with a new framework of modeling many impulsive effects caused by quantum effects of nanodevices.

Special Styles of the Book

In this book some special styles are designed to help the reader understand the text quickly and clearly. For theorems, lemmas, definitions and etc., the symbols ⊠ are used to terminate statements. For remarks, examples and proofs, the symbols ♦, ★ and ■ are used to show where are the ends of statements. Since many kinds of impulsive control systems were studied in this book, black boxes were used to highlight the control systems at the beginnings of corresponding sections. For readers who do not bother to read details of proofs, the highlighted control system models serve as an index for quick check of conditions for impulsive controller design for different plants. A special hypertext interface, called *reasoning flow chart* is used to represent the deriving processes of many proofs. A typical example is shown in (3.181). The following table lists some special symbols used in reasoning flow charts.

symbols	meanings	symbols	meanings
$A \Leftarrow B$	from B we have A;	$A \Rightarrow B$	from A we have B;
$\underbrace{A \leftrightsquigarrow B}_{C}$	$A \leftrightsquigarrow B$ is due to C;	$\begin{array}{c} B \\ \Uparrow\ A \end{array}$	from A we have B;
$\left.\begin{array}{c} A \\ B \end{array}\right\} \Rightarrow C$	from A and B we have C;	$\underbrace{B}_{A}$	from A we have B.

where the symbol $\leftrightsquigarrow$ denotes any of $<, >, \leq, \geq, \prec, \succ, \preceq, \succeq$, etc.

Organization of the Book

This book is organized in a highly self-contained and reader-friendly way. Many important theorems are accompanied by detailed proofs because these proofs can show the reader which assumption leads to which conclusion and therefore make it easy to understand, to apply and to improve these results. More important, the detailed proofs, which in many cases are constructive, can guide the design of impulsive controllers. Although the reader can browser papers and books to find many of these proofs, different symbols, jargons and typos contributed by different authors and publishers will make it a time-consuming and discouraging task.

In Chapter 1 the definitions of different kinds of impulsive control systems are presented. Some basic knowledge of impulsive differential equations such as the existence and continuations of solutions are introduced briefly. Explicit forms of solutions for different kinds of impulsive differential equations are presented. Some conditions for avoiding beating phenomena are also presented. Some extensively used definitions and mathematical results are summarized in this chapter.

In Chapter 2 we study time-invariant and time-varying linear impulsive control systems. The stability and controllability are presented.

In Chapter 3 the stability of impulsive control systems are studied based on comparison methods. We present the results based on single comparison system and multicomparison systems which can also be called as vector comparison system. The applications to impulsive control of chaotic systems are presented.

In Chapter 4 different methods for designing impulsive controllers with fixed-time impulses are presented. In this chapter the plants are nonlinear and in many cases we assume that they can be decomposed into linear parts and nonlinear parts. We also use Lyapunov second method to study the stability of impulsive control systems. The stability of sets and the stability in terms of two measures are also studied.

In Chapter 5 the impulsive control systems with impulses at variable time are studied by using linear decomposition methods and methods based on two measures. The stability of prescribed trajectories or control strategies is also studied.

In Chapter 6 the practical stability of impulsive control systems are studied by using different methods based on single comparison system, multicomparison system and two measures. The controllability in practical sense is also studied. We also study the practical stability of linear systems. Applications to impulsive control of nonautonomous chaotic systems are presented.

In Chapter 7 the partial stability of impulsive control systems with impulses at fixed-time and variable time is studied. We also study the stability of integro-differential impulsive control systems based on comparison methods, methods in terms of two measures and practical stability concept.

In Chapter 8 the principle of impulsive verb control are presented. Since verb control is a brand mew control paradigm, some basic knowledge of verb control is presented at the beginning of this chapter. Then the basic principles of verb impulsive control are presented. This chapter may of great interest to those readers who are familiar with fuzzy control because verb control is a natural extension of fuzzy control.

In Chapter 9 we study the impulsive control of periodic motions in linear periodic autonomous and nonautonomous systems. We also use parameter perturbation methods to control periodic motions in impulsive control systems. Applications to stepping motor control and impulsive control of chaotic systems to periodic motions are presented.

In Chapter 10 we present the impulsive control of almost periodic motions. We study two kinds of plants; namely, almost periodic plants and periodic plants driven by almost periodic control signals. The results can be used to control chaotic systems and design nanoelectronic circuits.

In Chapter 11 we present the applications of impulsive control theory to nanoelectronics which is an emerging discipline in electrical engineering. Although nanoelectronics is an extension of classical microelectronics, many device models in nanoelectronics are entirely different from those used in microelectronics because of the quantum mechanical effects of nanodevices. In this chapter we first present elementary impulsive device models that can be used to model different kinds of devices used in nanoelectronics. Some examples of nanodevices are then modeled by impulsive device models. Since one promising method to implement Boolean logic in nanoelectronic circuits is to encode two digit bits; namely, 0 or 1 by using different phase information of a periodic solution, the existence and stability of periodic solutions are of great interest to the design of nanoelectronic circuits. In this chapter, we will study the stability of periodic solutions of different kinds of nanoelectronic circuits.

Since the restriction to the volume of this book, it is impossible to include all aspects of impulsive systems and control. Therefore, some theoretical aspects such as global stability, absolute stability, optimal impulsive control and some application aspects such as impulsive secure and spread spectrum communications will be put in the second one of the two sister books [44].

Berkeley, California, June 2000

Tao Yang (杨涛)

Notation

$\triangleq$	defined as
$\mathbb{N}$	set of all positive integers
$\mathbb{Z}$	set of all integers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	$[0, +\infty)$
$\mathbb{C}$	set of all complex numbers
$\mathbb{R}^n$	real n-dimensional Euclidean space
$\mathbb{C}^n$	complex n-dimensional Euclidean space
$\mathrm{Re}(z)$	real part of the complex number z
z^*	complex conjugate of the complex number z
$\boldsymbol{x}$	$\mathrm{col}(x_1, x_2, \cdots, x_n)$; namely, element of $\mathbb{R}^n$ or $\mathbb{C}^n$
$\Vert\boldsymbol{x}\Vert$	norm of the element $\boldsymbol{x} \in \mathbb{R}$ or $\boldsymbol{x} \in \mathbb{C}$
$\Vert\boldsymbol{x}\Vert_2$	$\Vert\boldsymbol{x}\Vert_2 \triangleq \sqrt{\sum_{i=1}^{n} x_i x_i^*}$
$\overline{S}$,$[S]^c$	closure of the set S
I	$n \times n$ identity matrix
$I_m, I_{m\times m}$	$m \times m$ identity matrix
$A = A(a_{ij})$	$n \times n$ matrix with entries a_{ij}
$A = A(a_{ij})_{m\times m}$	$m \times m$ matrix with entries a_{ij}
A^{-1}	inverse of the matrix A
$A^{\dagger}$	pseudo-inverse of the matrix A
$A^{\top} = A(a_{ji})$	transposed matrix of the matrix $A = (a_{ij})$
$\det A$	determinant of the matrix A
$\mathrm{diag}(A_1, A_2, \cdots, A_m)$	block-diagonal matrix with blocks $A_1, A_2, \cdots, A_m$
$S^{\perp}$	the orthogonal complement of S
$\mathcal{H}_0$	the set of function $h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ satisfying $\inf_{\boldsymbol{x}\in\mathbb{R}^n} h(t, \boldsymbol{x}) = 0$ for each $t \in \mathbb{R}_+$

$\mathcal{H}$	the set of function $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $\inf_{(t,\boldsymbol{x})\in(\mathbb{R}_+,\mathbb{R}^n)} h(t,\boldsymbol{x}) = 0$, h is continuous in $(\tau_{i-1},\tau_i)\times\mathbb{R}^n$ and for each $\boldsymbol{x}\in\mathbb{R}^n, i = 1,2,\cdots$, $\lim_{(t,\boldsymbol{y})\to(\tau_i^+,\boldsymbol{x})} h(t,\boldsymbol{y}) = h(\tau_k^+,\boldsymbol{x})$ exists.
$\mathcal{K}$	the set of functions $\kappa \in C[\mathbb{R}_+,\mathbb{R}_+]$ such that $\kappa(t)$ is monotone strictly increasing and $\kappa(0) = 0$.
$\mathcal{K}[\Omega,\mathbb{R}_+]$	the set of functions $\kappa \in C[\Omega,\mathbb{R}_+]$ such that κ is monotone strictly increasing and $\kappa(0) = 0$.
$\mathcal{K}_1$	the set of functions $\kappa \in C[\mathbb{R}_+,\mathbb{R}_+]$ such that κ is increasing, $\kappa(0) = 0$ and $\lim_{w\to\infty}\kappa(w) = \infty$.
$\mathcal{K}_2$	the set of functions $\kappa : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(t,\cdot)\in\mathcal{K}_1$ for any fixed $t\in\mathbb{R}_+$.
$\mathcal{KR}$	the set of functions $\kappa\in\mathcal{K}$ satisfying $\lim_{t\to+\infty}\kappa(t) = \infty$.
$\mathcal{CK}$	the set of functions $\kappa \in C[\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+]$ satisfying $\kappa(t,s)\in\mathcal{K}$ for each $t\in\mathbb{R}_+$.
$\mathcal{PC}[D,F]$	the set of functions $\psi : D \to F$ which are continuous for $t\in D, t\neq\tau_k$, have discontinuities of first kind at the points τ_k and are left continuous.
$\mathcal{PC}^+[\mathbb{R}_+,\mathbb{R}_+]$	the set of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which are continuous for $t\in(\tau_{i-1},\tau_i]$ and $\lim_{t\to\tau_i^+}\psi(t) = \psi(\tau_i^+)$ exists, where $0<\tau_1<\tau_2<\cdots<\tau_i<\cdots, \lim_{\tau_i\to\infty}\tau_i = \infty$.
$\mathfrak{J}^+(t_0,\boldsymbol{x}_0)$	the maximal interval of the form (t_0,ω) in which the solution $\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ is defined.
$\mathfrak{J}^-(t_0,\boldsymbol{x}_0)$	the maximal interval of the form (α,t_0) in which the solution $\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ is defined.
$\mathcal{S}_\rho$	$\mathcal{S}_\rho \triangleq \{\boldsymbol{x}\in\mathbb{R}^n \mid \Vert\boldsymbol{x}\Vert<\rho\}$
$\overline{\mathcal{S}}_\rho$	$\overline{\mathcal{S}}_\rho \triangleq \{\boldsymbol{x}\in\mathbb{R}^n \mid \Vert\boldsymbol{x}\Vert\le\rho\}$
$\mathcal{S}_\rho(h)$	$\mathcal{S}_\rho(h) \triangleq \{(t,\boldsymbol{x})\in\mathbb{R}_+\times\mathbb{R}^n \mid h(t,\boldsymbol{x})<\rho,\quad h\in\mathcal{H}\}$
$\mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$	$\mathcal{S}_\rho^{\boldsymbol{x}_a(t)} \triangleq \{\boldsymbol{x}\in\mathbb{R}^n \mid \Vert\boldsymbol{x}-\boldsymbol{x}_a(t)\Vert<\rho,\quad \boldsymbol{x}_a(t)\in\mathbb{R}^n\}$
$\mathfrak{G}_i$	$\mathfrak{G}_i \triangleq \{(t,\boldsymbol{x}(t))\in\mathbb{R}_+\times\mathbb{R}^n \mid \tau_{i-1}(\boldsymbol{x})<t<\tau_i(\boldsymbol{x})\}, i = 1,2,\cdots,$ where $\tau_0(\boldsymbol{x}) = 0$ is assumed.

$\mathfrak{G}$	$\mathfrak{G} = \bigcup_{i=1}^{\infty} \mathfrak{G}_i$
$\mathfrak{N}(t, t+T)$	the number of impulses in interval $[t, t+T)$.

Contents

1. Preliminaries

The basic mathematical tool for designing impulsive control systems is the theory of impulsive differential equations. There are many English books on impulsive differential equations [13, 2, 1, 27, 3, 25, 7]. In this chapter, we present some basic knowledge on impulsive control system and impulsive differential equations. Since many basic knowledge of impulsive differential equations are well-documented in these above-mentioned books, we will not give detailed proofs for many conclusions. The reader can refer to these books for the basic mathematical aspects of impulsive differential equations.

1.1 What is Impulsive Control ?

Impulsive control is a control paradigm based on impulsive differential equations. In an impulsive control system, a nonimpulsive plant should have at least one "impulsively" *changeable* state variable. An impulsive plant should have impulsive effects in at least one state variable. The boundary between impulsive plants and nonimpulsive plants is fuzzy. There exists and will exist no crisp standard to distinguish impulsive plants from conventional plants. Therefore, if an impulsive control model can be well-fit into experimental observations, then keep it. Or else, you may need to model the plant under different control paradigms. However, if the control actions are taken in a much shorter(see, this is again a fuzzy quantity) time period comparing with the time constant, or natural period of the plant, then you may need to take a look at the possibility of modeling your controller under impulsive control paradigm.

Definition 1.1.1. *Impulsive control*
Given a nonimpulsive plant $\mathcal{P}$ whose state variable is denoted by $\boldsymbol{x} \in \mathbb{R}^n$, a set of control instants $T = \{\tau_k\}$, $\tau_k \in \mathbb{R}$, $\tau_k < \tau_{k+1}$, $k = 1, 2, \cdots$, and control laws $U(k, \boldsymbol{x}) \in \mathbb{R}^n$, $k = 1, 2, \cdots$. At each τ_k, $\boldsymbol{x}$ is changed impulsively by $\boldsymbol{x}(\tau_k^+) = \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x})$ such that the output $\boldsymbol{y} = \boldsymbol{g}(t, \boldsymbol{x})$, $\boldsymbol{g} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{y} \in \mathbb{R}^m$, approaches a goal $\boldsymbol{y}^* \in \mathbb{R}^m$ as $k \to \infty$. If the plant $\mathcal{P}$ is impulsive, then the control law can be nonimpulsive. ⊠

Remark 1.1.1.

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 1−15, 2001.

1. At least one state variable in a plant $\mathcal{P}$ can be changed "instantaneously" to any value which is given by a control law. In this sense, not all physical systems can be controlled by impulsive control schemes. The plant $\mathcal{P}$ may also subject to nonimpulsive control in parallel.
2. We only need to change the *changeable state variables* at discrete instants called *control instants*. Control instants are not necessary to be equidistant. Control instants can also be chosen whenever some conditions are satisfied. Control instants are generated by control instant generating law.
3. The control law $U(k, \boldsymbol{x})$ is something related to the concept of "control input" in other control methods. However, in an impulsive control system, $U(k, \boldsymbol{x})$ gives a sudden change of $\boldsymbol{x}$ at instant τ_k.
4. The purpose of invention of impulsive control is not to compete with the other control schemes, on the contrary, impulsive control provides a new viewpoint when the plant has at least one changeable state variable or when the plant has impulsive effects.

♦

We then give some examples of plants whose state variables can be changed instantaneously.

Example 1.1.1. To control the population of a kind of insect by using its natural enemies, we can cultivate the natural enemies in a laboratory and then release them at some proper instants. In this sense, some state variables of this system can be changed instantaneously. ★

Example 1.1.2. To control a mechanical system with impacts, the impacts can cause "impulsive" changes of velocities which are state variables. ★

Example 1.1.3. In a financial system, we suppose that one state variable is the amount of money in a market and the other state-variables are saving rates of a central bank. As it always occurs, the former can be controlled to a desired value by changing the latter instantaneously. ★

Example 1.1.4. In a nanoscale electronic circuit consisting of single-electron tunneling junctions(SETJ), the electron tunneling effects can cause impulsive changes of charge in SETJ junction capacitors. Since the charge is the state variable of a SETJ capacitor, the quantum mechanical effects can be modeled by impulsive differential equations. ★

In many cases impulsive control can give an efficient way to deal with plants which cannot endure continuous control inputs. Or, in some applications it is impossible to provide continuous control inputs. Let us consider the following examples.

1. In some cases, the plant can not be controlled by using continuous control. For example, a government can not change savings rates of its central bank everyday. A deep-space spacecraft can not leave its engine on continuously if it has only limited fuel supply.

2. In some cases, impulsive control is more efficient. For example, suppose that the population of a kind of bacterium and the density of a bactericide are two state variables of a system. We can control the population by instantaneously changing the density of the bactericide without enhance the drug resistance of bacteria which may be caused by continuous control.
3. For plants that are impulsive in nature, impulsive control is the best choice if not the only one. Nanodevices are among this kind of plant.

1.2 Different Types of Impulsive Control Schemes

Impulsive control systems can be classified into the following three types based on the characteristics of plants and control laws.

A type-I impulsive control system is given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{y}), \quad t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{y} &= \boldsymbol{g}(t, \boldsymbol{x}) \end{aligned} \tag{1.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the state variable and the output, respectively. $U(k, \boldsymbol{y})$ is the impulsive control law. In this kind of system, the control input is implemented by the "sudden jumps" of some state variables.

A type-II impulsive control system is given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{y}), \quad t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{y} &= \boldsymbol{g}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad \tilde{\boldsymbol{u}} = \gamma(t, \boldsymbol{y}) \end{aligned} \tag{1.2}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the state variable and the output, respectively. $U(k, \boldsymbol{y})$ and $\tilde{\boldsymbol{u}}$ are the impulsive control law and the continuous control law, respectively. In this system, there are two kinds of control inputs. The first kind is the continuous control input $\tilde{\boldsymbol{u}}$ and the second one is the impulsive control input $U(k, \boldsymbol{y})$.

A type-III impulsive control system is given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= \boldsymbol{j}_k(\boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{y} &= \boldsymbol{g}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad \tilde{\boldsymbol{u}} = \gamma(t, \boldsymbol{y}) \end{aligned} \tag{1.3}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the state variable and the output, respectively. $\tilde{\boldsymbol{u}}$ is the continuous control law. In this kind of systems, the plant itself is impulsive and the control is continuous.

1.3 Mathematical Models of Systems with Impulsive Effects

The mathematical model of an evolving process with impulsive effects is given by

1. A differential equation
$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{1.4}$$
where $\boldsymbol{x} \in \Omega$ and $t \in \mathbb{R}$.
2. A *switching set* $\Sigma_t \in \Omega \times \mathbb{R}$.
3. A *jumping operator* $\mathcal{J}_t$ defined on the switching set Σ_t such that $\mathcal{J}_t \circ \Sigma_t \subset \Omega \times \mathbb{R}$.

Following a tradition from classical mechanics , the set Ω is called the *phase space*; namely, the set of all possible states of this evolving process. Phase space is also called *state space.* A *distance* d between two states should be defined, allowing for comparison between two states. Therefore, a phase space is a *metric space.* In this book, the phase space is chosen as a real vector space $\mathbb{R}^n$ of some finite dimension $n \in \mathbb{N}$, or a manifold in this space. We usually use the *Euclidean norm* to measure two states $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{< \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} >} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $< \cdot, \cdot >$ denotes the *scalar product* in $\mathbb{R}^n$. The *state variable* $\boldsymbol{x}(t)$ represents the state of the evolving process at time $t \in \mathbb{T}$. In general $\mathbb{T}$ can be any *ordered set.* In control literature, we usually consider two kinds of dynamical systems; namely, *continuous-time* dynamical systems with $\mathbb{T} = \mathbb{R}$ and *discrete-time* dynamical systems with $\mathbb{T} = \mathbb{Z}$. If we need only n parameters to describe $\boldsymbol{x}(t)$ at t and n is a finite number, then this evolving process is *finite dimensional.* In this case, $\boldsymbol{x}(t)$ is an n-dimensional vector of the Euclidean space $\mathbb{R}^n$ and Ω is a subset of $\mathbb{R}^n$. We call $\Omega \times \mathbb{R}$ the *extended phase space.*

With an *initial state*, $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$[1], (1.4) defines a *dynamical system* which is the mathematical formalization of *deterministic processes.* The solution to (1.4) is often written as $\phi_t(\boldsymbol{x}_0)$, or $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$. The map $\phi_t : \Omega \to \Omega$ is called the *evolution operator* of the dynamical system. The one-parameter family of mappings $\{\phi_t\}_{t \in \mathbb{T}}$ satisfies

$$\phi_{t+s} = \phi_t \circ \phi_s$$

[1] Note that in a conventional dynamical system, the initial state is defined as $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

and

$$\phi_0(\boldsymbol{x}) = \boldsymbol{x},$$

and is called the *flow*. The set of points $\{\phi_t(\boldsymbol{x}_0)|\ t \in \mathbb{T}\}$ is called a *trajectory*, or an *orbit* of (1.4) through $\boldsymbol{x}_0$. If an impulsive event happens at $t = \tau_i$, then the trajectory *hits* the switching set Σ_t at $t = \tau_i$; namely, $\boldsymbol{x}(\tau_i) \in \Sigma_t$. As soon as $\boldsymbol{x}(t)$ hits the switching set Σ_t at $t = \tau_i$, it "jumps" immediately to a point defined by the jumping operator $\mathcal{J}_t$; namely, $\boldsymbol{x}(\tau^+) = \mathcal{J}_t \circ \boldsymbol{x}(\tau)$. Thus, the solution of the impulsive system $\phi_t(\boldsymbol{x}_0)$, satisfies (1.4) outside the switching set Σ_t and has discontinuities of the first kind at the points where it hits the switching set Σ_t with the jumps

$$\Delta \boldsymbol{x}(t) = \phi_{t^+}(\boldsymbol{x}_0) - \phi_t(\boldsymbol{x}_0) = \mathcal{J}_t \circ \phi_t(\boldsymbol{x}_0) - \phi_t(\boldsymbol{x}_0). \tag{1.5}$$

We then have a concise form of impulsive differential equation as

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad (t, \boldsymbol{x}) \notin \Sigma_t, \\ \Delta \boldsymbol{x} &= \mathcal{J}_t \circ \boldsymbol{x} - \boldsymbol{x}, \qquad (t, \boldsymbol{x}) \in \Sigma_t. \end{aligned} \tag{1.6}$$

Equation (1.6) can have the following three kinds of solutions:

1. $\phi_t(\boldsymbol{x}_0)$ does not hit Σ_t or the hitting points are fixed points of $\mathcal{J}_t$. In this case, there is no impulsive event.
2. $\phi_t(\boldsymbol{x}_0)$ hits Σ_t at a finite number of points that are not fixed points of $\mathcal{J}_t$. In this case, there are finite number of impulsive events.
3. $\phi_t(\boldsymbol{x}_0)$ hits Σ_t in a countable number of points that are not fixed points of $\mathcal{J}_t$. In this case, there are countable number of impulsive events.

Based on different characteristics of impulsive events, we usually encounter three primary types of impulsive systems

1. in which impulses occur at fixed time;
2. in which impulses occur when the trajectory hits a hypersurface in the extended phase space;
3. which are discontinuous dynamical systems.

1.3.1 Impulsive Events at Fixed Time

If the impulsive events occur in a finite or infinite sequence of time $\{\tau_k\}$, then system (1.6) can be written as

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad t \neq \tau_k, \\ \Delta \boldsymbol{x} &= J_k(\boldsymbol{x}), \qquad t = \tau_k. \end{aligned} \tag{1.7}$$

The solution of (1.7) is a piecewise continuous function $\boldsymbol{x} = \phi(t)$ that has discontinuities of the first kind at $t = \tau_k$. $\frac{d\phi(t)}{dt} = \boldsymbol{f}(t, \phi(t))$ is satisfied for all $t \neq \tau_k$. At $t = \tau_k$, $\phi(t)$ satisfies the following jumping condition:

$$\Delta \phi|_{t=\tau_k} = \phi(\tau_k^+) - \phi(\tau_k) = J_k(\phi(\tau_k)). \tag{1.8}$$

Remark 1.3.1. In some references (1.8) is written as

$$\Delta\phi|_{t=\tau_k} = \phi(\tau_k^+) - \phi(\tau_k^-) = J_k(\phi(\tau_k^-)) \tag{1.9}$$

where $\phi(\tau_k^-) = \phi(\tau_k)$ is assumed. ♦

1.3.2 Impulsive Events at Variable Time

In this kind of system, the impulsive events occurs when trajectories hit a hypersurface $\aleph(t, \boldsymbol{x}) = 0$. This kind of impulsive differential equation can be written as

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad \aleph(t, \boldsymbol{x}) \neq 0, \\ \Delta\boldsymbol{x} &= J(t, \boldsymbol{x}), \qquad \aleph(t, \boldsymbol{x}) = 0. \end{aligned} \tag{1.10}$$

In this case, the switching set is given by

$$\Sigma_t = \{(t, \boldsymbol{x}) |\ \aleph(t, \boldsymbol{x}) = 0\}, \tag{1.11}$$

and the jumping operator $\mathcal{J}_t$ is given by

$$\mathcal{J}_t : (t, \boldsymbol{x}) \to (t, \boldsymbol{x} + J(t, \boldsymbol{x})). \tag{1.12}$$

If the equation $\aleph(t, \boldsymbol{x}) = 0$ has a countable number of solutions with respect to t, we denote these solutions by $t = \tau_k(\boldsymbol{x})$ and index then by the set of integers (or a subset of integers) such that $\tau_k(\boldsymbol{x}) \to \infty$ as $k \to \infty$ and $\tau_k(\boldsymbol{x}) \to -\infty$ as $k \to -\infty$. In this case, the jumping operator is given by

$$\mathcal{J}_{\tau_k(\boldsymbol{x})} : \boldsymbol{x} \to \boldsymbol{x} + J(\tau_k(\boldsymbol{x}), \boldsymbol{x}), \tag{1.13}$$

which can be simplified as

$$\mathcal{J}_{\tau_k(\boldsymbol{x})} : \boldsymbol{x} \to \boldsymbol{x} + J_k(\boldsymbol{x}). \tag{1.14}$$

Then (1.10) can be rewritten as

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad t \neq \tau_k(\boldsymbol{x}), \\ \Delta\boldsymbol{x} &= J_k(\boldsymbol{x}), \qquad t = \tau_k(\boldsymbol{x}). \end{aligned} \tag{1.15}$$

System (1.10) is more difficult to study than system (1.7). The solutions of system (1.10) starting at different initial conditions have different points of discontinuity. A solution of system (1.10) may hit the same hypersurface $\aleph(t, \boldsymbol{x}) = 0$ many times and cause so called "*beating phenomenon*" or "*pulse phenomenon*". Different solutions of system (1.10) may coincide after some time and behave like a single solution thereafter and thus cause "*confluence*".

1.3.3 Discontinuous Dynamical Systems

When the differential equation (1.4) does not explicitly depend on t and the jumping operator $\mathcal{J}_t = \mathcal{J}$ for all $t \in \mathbb{R}$, let $\mathcal{J} : \Sigma \to \Sigma_0$ be a mapping between the switching set Σ and a target set Σ_0, then a discontinuous dynamical system is given by

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \notin \Sigma, \\ \Delta \boldsymbol{x} &= \mathcal{J}\boldsymbol{x} - \boldsymbol{x} = J(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Sigma. \end{aligned} \tag{1.16}$$

Many electrical circuit models containing switching capacitors belong to this kind of impulsive system. To get interesting phenomena from this kind of impulsive system, we usually assume that the switching set Σ is a compact or locally compact manifold in the phase space with dimension $(n-1)$. Or else, there is a good chance that a large part of the motions will not have impulsive effects.

1.4 Existence and Continuation of Solutions

Consider the *initial value problem* for the following impulsive differential equation:

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= J_k(\boldsymbol{x}), \qquad t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \qquad t_0 \geq 0 \end{aligned} \tag{1.17}$$

where $\boldsymbol{f} : D \to \mathbb{R}^n$ and $J_k : \Omega \to \mathbb{R}^n$. Here we let $\Omega \subset \mathbb{R}^n$ be an open set and $D = \mathbb{R}_+ \times \Omega$. We assume that $\tau_k \in C[\Omega, \mathbb{R}_+]$, $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ and $\lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty$ for $k = 1, 2, \cdots$, $\boldsymbol{x} \in \Omega$.

Definition 1.4.1. *A function $\boldsymbol{x} : (t_0, t_0 + \varepsilon) \to \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$, $\varepsilon > 0$, is a solution of (1.17) if*

1. *$\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$ and $(t, \boldsymbol{x}(t)) \in D$ for $t \in [t_0, t_0 + \varepsilon)$,*
2. *$\boldsymbol{x}(t)$ is continuously differentiable and satisfies $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t))$ for $t \in [t_0, t_0 + \varepsilon)$ and $t \neq \tau_k(\boldsymbol{x}(t))$,*
3. *if $t \in [t_0, t_0 + \varepsilon)$ and $t = \tau_k(\boldsymbol{x}(t))$, then $\boldsymbol{x}(t^+) = \boldsymbol{x}(t) + J_k(\boldsymbol{x})$, and at such t's we always assume that $\boldsymbol{x}(t)$ is left continuous and $\alpha \neq \tau_i(\boldsymbol{x}(\alpha))$ for any $i = 1, 2, \cdots$, $t < \alpha < \delta$, for some $\delta > 0$.*

⊠

Remark 1.4.1. Observe that instead of the initial condition, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, used in ordinary differential equations, the limiting condition $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$ is used.

This is because $(t_0, \boldsymbol{x}_0)$ may be such that $t_0 = \tau_k(\boldsymbol{x}_0)$ for some k. Whenever $t_0 \neq \tau_k(\boldsymbol{x}_0)$ for any k, then $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$ is the same as $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Unlike ordinary differential equations, (1.17) may have no solution event if $\boldsymbol{f}$ is continuous or continuously differentiable since its only solution of the initial value problem may entirely lie on a surface $\Sigma_k : t = \tau_k(\boldsymbol{x})$. Thus we need some extra condition on τ_k and/or $\boldsymbol{f}$ besides continuity in order to establish existence theory for (1.17). ♦

Theorem 1.4.1. *Assume that*

1. *$\boldsymbol{f} : D \to \mathbb{R}^n$ is continuous at $t \neq \tau_k(\boldsymbol{x})$, $k = 1, 2, \cdots$, and for each $(t, \boldsymbol{x}) \in D$ there is an $l \in \mathcal{L}^1_{\text{loc}}$ such that for any $(\alpha, \boldsymbol{y})$ in a small neighborhood of $(t, \boldsymbol{x})$ we have*
$$\|\boldsymbol{f}(\alpha, \boldsymbol{y})\| \leq l(\alpha); \tag{1.18}$$
2. *for any k, $t_1 = \tau_k(\boldsymbol{x}_1)$ implies the existence of a $\delta > 0$ such that*
$$t \neq \tau_k(\boldsymbol{x}) \tag{1.19}$$
for any $t - t_1 \in (0, \delta)$ and $\|\boldsymbol{x} - \boldsymbol{x}_1\| < \delta$.

Then, for each $(t_0, \boldsymbol{x}_0) \in D$, there exists a solution $\boldsymbol{x} : [t_0, t_0 + \gamma) \to \mathbb{R}^n$ of the initial value problem (1.17) for some $\gamma > 0$. ⊠

Proof. If $t_0 \neq \tau_k(\boldsymbol{x}_0)$ for any k then the conclusion is true. If $t_0 = \tau_k(\boldsymbol{x}_0)$ for some k, from (1.18) and since $\boldsymbol{f}$ is continuous, we know that there exists a local solution $\boldsymbol{x}(t)$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ and $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$. Since $\tau_i(\boldsymbol{x}) < \tau_j(\boldsymbol{x})$ for $i < j$ and $t_0 = \tau_k(\boldsymbol{x}_0)$, we know $t \neq \tau_j(\boldsymbol{x})$ for $j \neq k$ and t sufficiently close to t_0. Condition (1.19) does not permit $t = \tau_k(\boldsymbol{x}(t))$ in a sufficient small right neighborhood of t_0. Hence $\boldsymbol{x}(t)$ is a local solution of system (1.17). ■

Theorem 1.4.2. *Assume that*

1. *$\boldsymbol{f} : D \to \mathbb{R}^n$ is continuous;*
2. *$\tau_k : \Omega \to (0, \infty)$ are differentiable;*
3. *if $t_1 = \tau_k(\boldsymbol{x}_1)$ for some $(t_1, \boldsymbol{x}_1) \in D$ and $k \geq 1$, then there is a $\delta > 0$ such that*
$$\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}) \neq 1 \tag{1.20}$$
for $(t, \boldsymbol{x}) \in D$ with $t - t_1 \in (0, \delta)$ and $|\boldsymbol{x} - \boldsymbol{x}_1| < \delta$.

Then, for each $(t_0, \boldsymbol{x}_0) \in D$, there exists a solution $\boldsymbol{x} : [t_0, t_0 + \alpha) \to \mathbb{R}^n$ of the initial value problem (1.17) for some $\alpha > 0$. ⊠

Proof. For the cases of $t_0 \neq \tau_k(\boldsymbol{x}_0)$ for any k the proof is straightforward. If $t_0 = \tau_k(\boldsymbol{x}_0)$ for some $k \geq 1$ and $\boldsymbol{x}(t)$ is a solution of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ and $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$, set $\gamma(t) = t - \tau_k(\boldsymbol{x}(t))$. Then $\gamma(t_0) = 0$ and

$$\dot{\gamma}(t) = 1 - \frac{\partial \tau_k(\boldsymbol{x}(t))}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}(t)). \tag{1.21}$$

From (1.20) it follows that $\dot{\gamma}(t) \neq 0$ in a small right neighborhood of t_0. Hence $\gamma(t)$ is either strictly increasing or decreasing in that neighborhood and we have $t \neq \tau_k(\boldsymbol{x}(t))$ for $t - t_0 \in (0, \delta)$ for some $\delta > 0$. The rest of proof is similar to that of Theorem 1.4.1. ■

We then consider the continuation of solutions of (1.17) to the right. Given a solution $\boldsymbol{x}(t)$ of (1.17) defined on $[t_0, t_0 + \alpha)$ with $\alpha > 0$, we say that a solution $\boldsymbol{x}^*(t)$ of (1.17) is a *proper* continuation to the right of $\boldsymbol{x}(t)$ if $\boldsymbol{x}^*(t)$ is defined on $[t_0, t_0 + \beta)$ for some $\beta > \alpha$ and $\boldsymbol{x}(t) = \boldsymbol{x}^*(t)$ for $t \in [t_0, t_0 + \alpha)$. The interval $[t_0, t_0 + \alpha)$ is called the *maximal interval of existence* of a solution $\boldsymbol{x}(t)$ of (1.17), if $\boldsymbol{x}(t)$ is well defined on $[t_0, t_0 + \alpha)$ and it does not have any proper continuation to the right. We then have the following theorem.

Theorem 1.4.3. *Assume that* $\Omega = \mathbb{R}^n$ *and*

1. $\boldsymbol{f} : D \to \mathbb{R}^n$ *is continuous;*
2. $J_k \in C[\Omega, \mathbb{R}^n]$, $\tau_k \in C[\Omega, \mathbb{R}_+]$ *for all* $k > 1$.

Then, for any solution $\boldsymbol{x}(t)$ *of (1.17) with a finite* $[t_0, \beta)$ *as its maximal interval of existence, we have*

$$\lim_{t \to \beta^-} \|\boldsymbol{x}(t)\| = \infty, \tag{1.22}$$

if one of the following three conditions is satisfied:

1. *for any* $k \geq 1$, $t_1 = \tau_k(\boldsymbol{x}_1)$ *implies the existence of a* $\delta > 0$ *such that* $t \neq \tau_k(\boldsymbol{x})$ *for all* $(t, \boldsymbol{x})$ *with* $t - t_1 \in (0, \delta)$ *and* $\|\boldsymbol{x} - \boldsymbol{x}_1\| < \delta$;
2. *for all* $k \geq 1$, $t_1 = \tau_k(\boldsymbol{x}_1)$ *implies that* $t_1 \neq \tau_j(\boldsymbol{x}_1 + J_k(\boldsymbol{x}_1))$ *for all* $j \geq 1$;
3. $\tau_k \in C^1[\Omega, \mathbb{R}_+]$ *for all* $k \geq 1$ *and* $t_1 = \tau_k(\boldsymbol{x}_1)$ *implies that* $t_1 = \tau_j(\boldsymbol{x}_1 + J_k(\boldsymbol{x}_1))$ *for some* $j \geq 1$ *and*

$$\frac{\partial \tau_j(\boldsymbol{x}_1^+)}{\partial \boldsymbol{x}} \boldsymbol{f}(t_1, \boldsymbol{x}_1^+) \neq 1, \tag{1.23}$$

where $\boldsymbol{x}_1^+ = \boldsymbol{x}_1 + J_k(\boldsymbol{x}_1)$.

⊠

Proof. See pages 17 to 20 of [13]. ■

Some other results on the continuation of solutions can be found in Chapter 3 of [2]. We do not repeat them here.

1.5 Beating Phenomena

The solutions of an impulsive differential equation may hit the same switching surface finite or infinite number of times causing *beating phenomena*. Consider the following impulsive differential equation:

$$\begin{aligned}
\frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\
\Delta \boldsymbol{x} &= J_k(\boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \\
\boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad t_0 \geq 0
\end{aligned} \tag{1.24}$$

where $\boldsymbol{f} \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^n]$ and $\Omega \subset \mathbb{R}^n$ is an open set.

Theorem 1.5.1. *Assume that*

1. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^n]$, $\tau_k \in C^1[\Omega, \mathbb{R}_+]$, $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ *for every* k, $\lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty$ *uniformly in* $\boldsymbol{x} \in \Omega$ *and* $J_k \in [\Omega, \mathbb{R}^n]$;
2. *a)* $\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}) \leq 0$ *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Omega$, *and*
   b) $\boldsymbol{x} + J_k(\boldsymbol{x}) \in \Omega$ *for* $\boldsymbol{x} \in \Omega$ *and*

$$\left(\frac{\partial \tau_k(\boldsymbol{x} + \epsilon J_k(\boldsymbol{x}))}{\partial \boldsymbol{x}} \right) J_k(\boldsymbol{x}) \leq 0,$$

$\epsilon \in [0, 1]$, *for every* k.

Then, each solution of (1.24) hits any given switching surface $\Sigma_i : t = \tau_i(\boldsymbol{x})$ *at most once.* ⊠

Proof. See pages 22 and 23 of [13]. ■

Theorem 1.5.2. *Assume that*

1. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^n]$, $\tau_k \in C^1[\Omega, \mathbb{R}_+]$, $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ *for every* k, $\lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty$ *uniformly in* $\boldsymbol{x} \in \Omega$ *and* $J_k \in [\Omega, \mathbb{R}^n]$;
2. *a)* $\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}) \leq \delta$, $0 \leq \delta \leq 1$, *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Omega$;
   b) $\boldsymbol{x} + J_k(\boldsymbol{x}) \in \Omega$ *for* $\boldsymbol{x} \in \Omega$ *and*

$$\left(\frac{\partial \tau_k(\boldsymbol{x} + \epsilon J_k(\boldsymbol{x}))}{\partial \boldsymbol{x}} \right) J_k(\boldsymbol{x}) < 0,$$

$\epsilon \in [0, 1]$, *for every* k.

Then, each solution of (1.24) hits any given switching surface $\Sigma_i : t = \tau_i(\boldsymbol{x})$ *at most once.* ⊠

Proof. Similar to that of Theorem 1.5.1. ■

Theorem 1.5.3. *Assume that*

1. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^n]$, $J_k \in C[\Omega, \mathbb{R}^n]$, $\tau_k \in C^1[\Omega, \mathbb{R}_+]$, $\tau_k(\boldsymbol{x})$ *is bounded and* $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ *for every* k;
2. *a)* $\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}) \leq 1$, *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Omega$;
   b)

$$\left(\frac{\partial \tau_k(\boldsymbol{x} + \epsilon J_k(\boldsymbol{x}))}{\partial \boldsymbol{x}} \right) J_k(\boldsymbol{x}) < 0;$$

c)

$$\left(\frac{\partial \tau_k(\boldsymbol{x} + \epsilon J_{k-1}(\boldsymbol{x}))}{\partial \boldsymbol{x}}\right) J_{k-1}(\boldsymbol{x}) \geq 0$$

where $\boldsymbol{x} + J_k(\boldsymbol{x}) \in \Omega$ *for* $\boldsymbol{x} \in \Omega$ *and* $\epsilon \in [0, 1]$*, for every* k*.*

Then, each solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ *of (1.24) with* $0 \leq t_0 < \tau_1(\boldsymbol{x}_0)$ *hits each switching surface* $\Sigma_i : t = \tau_i(\boldsymbol{x})$ *exactly once.* ⊠

Proof. See pages 24 and 25 of [13]. ■

1.6 Solutions of Impulsive Differential Equations

Let us consider the solution of the following linear impulsive differential equation:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x}, \ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x}, \ t = \tau_k, \quad k = 1, 2, \cdots. \end{cases} \tag{1.25}$$

Theorem 1.6.1. *Assume that*

1.

$$0 = \tau_0 < \tau_1 < \cdots, \quad \lim_{k \to \infty} \tau_k = \infty; \tag{1.26}$$

2.

$$A \in \mathcal{PC}[\mathbb{R}^+, \mathbb{C}^{n \times n}], \quad B_k \in \mathbb{C}^{n \times n}, \quad k = 1, 2, \cdots. \tag{1.27}$$

Then for any $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ *there exists a unique solution* $\phi(t)$ *of system (1.25) which is defined for* $t > t_0$ *with* $\phi(t_0^+) = \boldsymbol{x}_0$*. Furthermore, if* $\det(I + B_k) \neq 0$ *for every* k*, then this solution is defined for all* $t \in \mathbb{R}_+$*.* ⊠

Proof. See the proof of Theorem 3.6 of [2]. ■

Assume that $\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$ and let the following n functions

$$\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_n(t) \tag{1.28}$$

be solutions of system (1.25) defined in $\mathbb{R}$. Let us set

$$\Phi(t) \triangleq (\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_n(t)) \tag{1.29}$$

as a matrix function with columns as the solutions in (1.28). If $\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_n(t)$ are linearly independent, then we call $\Phi(t)$ a *fundamental matrix* of system (1.25). $\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_n(t)$ are linearly independent in $\mathbb{R}$ if and only if $\det \Phi(t_0^+) \neq 0$ for some $t_0 \in \mathbb{R}$.

The following facts about fundamental matrices are useful.

Theorem 1.6.2. *Assume that*

1. *$\tau_k < \tau_{k+1}$, $k \in \mathbb{Z}$ and*

$$\lim_{k=-\infty} \tau_k = -\infty, \quad \lim_{k=\infty} \tau_k = \infty;$$

2. *$A \in \mathcal{PC}[\mathbb{R}, \mathbb{C}^{n\times n}]$ and $B_k \in \mathbb{C}^{n\times n}$ for all $k \in \mathbb{Z}$;*
3. *$\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$ and $\Phi(t)$ is a fundamental matrix of system (1.25).*

Then we have the following two conclusions:

1. *for any constant matrix $A \in \mathbb{C}^{n\times n}$, $\Phi(t)A$ is a solution of system (1.25);*
2. *If $\Phi_1 : \mathbb{R} \to \mathbb{C}^{n\times n}$ is a solution of system (1.25), then there is a unique matrix $S \in \mathbb{C}^{n\times n}$ such that $\Phi_1(t) = \Phi(t)S$. Furthermore, if $\Phi_1(t)$ is also a fundamental matrix of system (1.25), then* $\det S \neq 0$.

⊠

Let the *state transition matrix* (or, fundamental matrix) of the linear system

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}, \quad t \in (\tau_{k-1}, \tau_k] \tag{1.30}$$

be $\Phi_k(t, s)$ $(t, s \in (\tau_{k-1}, \tau_k])$ and conditions (1.26) and (1.27) hold, then the solution of system (1.25) is given by

$$\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) = \Psi(t, t_0^+)\boldsymbol{x}_0, \tag{1.31}$$

where $\Psi(t, s)$ is called *state transition matrix* (or, fundamental matrix, Cauchy matrix) of system (1.25) and given by

$$\Psi(t,s) = \begin{cases} \Phi_k(t,s), \\ \qquad\qquad \text{for } t, s \in (\tau_{k-1}, \tau_k], \\ \Phi_{k+1}(t, \tau_k^+)(I + B_k)\Phi_k(\tau_k, s), \\ \qquad\qquad \text{for } \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}, \\ \Phi_k(t, \tau_k)(I + B_k)^{-1}\Phi_{k+1}(\tau_k^+, s), \\ \qquad\qquad \text{for } \tau_{k-1} < t \leq \tau_k < s \leq \tau_{k+1}, \\ \Phi_{k+1}(t, \tau_k^+)\left(\prod_{j=k}^{i+1}(I + B_j)\Phi_j(\tau_j, \tau_{j-1}^+)\right) \\ \quad (I + B_i)\Phi_i(\tau_i, s), \\ \qquad\qquad \text{for } \tau_{i-1} < s \leq \tau_i < \tau_k < t \leq \tau_{k+1}, \\ \Phi_i(t, \tau_i)\left(\prod_{j=i}^{k-1}(I + B_j)^{-1}\Phi_{j+1}(\tau_j^+, \tau_{j+1})\right) \\ \quad (I + B_k)^{-1}\Phi_{k+1}(\tau_k^+, s), \\ \qquad\qquad \text{for } \tau_{i-1} < t \leq \tau_i < \tau_k < s \leq \tau_{k+1}. \end{cases} \tag{1.32}$$

$\Psi(t, s)$ has the following properties:

$$\begin{aligned} \Psi(t,t) &= I, \\ \Psi(\tau_k^-, \tau_k) &= \Psi(\tau_k, \tau_k^-) = I, \\ \Psi(\tau_k^+, s) &= (I + B_k)\Psi(\tau_k, s), \\ \Psi(s, \tau_k^+) &= (I + B_k)^{-1}\Psi(s, \tau_k), \quad s \neq \tau_k^+, \\ \frac{\partial \Psi}{\partial t}(t,s) &= A(t)\Psi(t,s), \quad t \neq \tau_k. \end{aligned} \tag{1.33}$$

We then consider the solution of the following linear impulsive nonautonomous system:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{g}(t),\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x} + \boldsymbol{j}_k, \quad t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \tag{1.34}$$

where $\boldsymbol{g} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{C}^n]$, $\boldsymbol{j}_k \in \mathbb{C}^n$. Then the solution of system (1.34), $\boldsymbol{x}(t) \triangleq \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, is given by

$$\boldsymbol{x}(t) = \begin{cases} \Psi(t, t_0^+)\boldsymbol{x}(t_0^+) + \int_{t_0}^t \Psi(t, s)\boldsymbol{g}(s)ds \\ \qquad + \sum_{t_0 < \tau_k < t} \Psi(t, \tau_k^+)\boldsymbol{j}_k, & t > t_0, \\ \Psi(t, t_0^+)\boldsymbol{x}(t_0^+) + \int_{t_0}^t \Psi(t, s)\boldsymbol{g}(s)ds \\ \qquad - \sum_{t \leq \tau_k \leq t_0} \Psi(t, \tau_k^+)\boldsymbol{j}_k, & t \leq t_0. \end{cases} \tag{1.35}$$

We then consider the solution of the following impulsive differential equations:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}),\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x} + \boldsymbol{j}_k(\boldsymbol{x}),\ t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \tag{1.36}$$

where $A \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^{n\times n}]$, $B_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{g} \in \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$, and $\boldsymbol{j}_k \in \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^n$. Then the solution of system (1.36), $\boldsymbol{x}(t) \triangleq \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, is given by

$$\boldsymbol{x}(t) = \begin{cases} \Psi(t, t_0^+)\boldsymbol{x}(t_0^+) + \int_{t_0}^t \Psi(t, s)\boldsymbol{g}(s, \boldsymbol{x}(s))ds \\ \qquad + \sum_{t_0 < \tau_k < t} \Psi(t, \tau_k^+)\boldsymbol{j}_k(\boldsymbol{x}(\tau_k)), & t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0), \\ \Psi(t, t_0^+)\boldsymbol{x}(t_0^+) + \int_{t_0}^t \Psi(t, s)\boldsymbol{g}(s, \boldsymbol{x}(s))ds \\ \qquad - \sum_{t \leq \tau_k \leq t_0} \Psi(t, \tau_k^+)\boldsymbol{j}_k(\boldsymbol{x}(\tau_k)), & t \in \mathfrak{J}^-(t_0, \boldsymbol{x}_0). \end{cases} \tag{1.37}$$

1.7 Definitions and Basics

Definition 1.7.1. *Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, then V is said to belong to class $\mathcal{V}_0$ if*

1. *V is continuous in $(\tau_{k-1}, \tau_k] \times \mathbb{R}^n$ and for each $\boldsymbol{x} \in \mathbb{R}^n$, $k = 1, 2, \cdots$,*

$$\lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} V(t, \boldsymbol{y}) = V(\tau_k^+, \boldsymbol{x}) \tag{1.38}$$

   exists;
2. *V is locally Lipschitzian in $\boldsymbol{x}$ and $V(t, 0) = 0$ for all $t \in \mathbb{R}_+$.*

$\boxtimes$

Definition 1.7.2. *Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, then V is said to belong to class $\mathcal{V}_1$ if $V \in \mathcal{V}_0$ and it is continuously differentiable on $\mathfrak{G}$.* $\boxtimes$

Definition 1.7.3. *A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a member of class $\mathcal{V}_2$ if $V(t, \boldsymbol{x})$ is continuous on $\mathfrak{G}_i$ and for $(t_0, \boldsymbol{x}_0) \in \Sigma_i$, where $\Sigma_i : t = \tau_i(\boldsymbol{x})$ is a switching surface, the following limits exist*

$$\lim_{(t,\boldsymbol{x})\to(t_0,\boldsymbol{x}_0),(t,\boldsymbol{x})\in\mathfrak{G}_i} V(t, \boldsymbol{x}) = V(t_0^-, \boldsymbol{x}),$$
$$\lim_{(t,\boldsymbol{x})\to(t_0,\boldsymbol{x}_0),(t,\boldsymbol{x})\in\mathfrak{G}_{i+1}} V(t, \boldsymbol{x}) = V(t_0^+, \boldsymbol{x}), \tag{1.39}$$

and $V(t_0^-, \boldsymbol{x}) = V(t_0^+, \boldsymbol{x})$ holds. ⊠

Definition 1.7.4. *For two n-vectors $\boldsymbol{x}^\top = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y}^\top = (y_1, y_2, \cdots, y_n)$ we define the following componentwise inequalities:*

1. *$\boldsymbol{x} \prec \boldsymbol{y}$ if $x_i < y_i$ for all $i = 1, 2, \cdots, n$;*
2. *$\boldsymbol{x} \succ \boldsymbol{y}$ if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;*
3. *$\boldsymbol{x} \preceq \boldsymbol{y}$ if $x_i = y_i$ for at least one $i \in \{1, 2, \cdots, n\}$ and $x_i < y_i$ for the rest $i \in \{1, 2, \cdots, n\}$;*
4. *$\boldsymbol{x} \succeq \boldsymbol{y}$ if $x_i = y_i$ for at least one $i \in \{1, 2, \cdots, n\}$ and $x_i > y_i$ for the rest $i \in \{1, 2, \cdots, n\}$.*

⊠

Definition 1.7.5. *A function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is quasimonotone nondecreasing if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $x_i = y_i$ for some $i \in \{1, 2, \cdots, n\}$, then $f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{y})$ for all $i \in \{1, 2, \cdots, n\}$.* ⊠

Definition 1.7.6. *A function $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be nondecreasing in $\mathbb{R}^n$ if for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u} \preceq \boldsymbol{v}$ implies $\boldsymbol{g}(\boldsymbol{u}) \preceq \boldsymbol{g}(\boldsymbol{v})$.* ⊠

Let a general impulsive system be

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}), \quad t \neq \tau_k, \quad t \in J, \\ \mathbf{x}(\tau_k^+) &= \Psi_k(\mathbf{x}(\tau_k)), \quad k = 1, \cdots, q, \\ \mathbf{x}(0) &= \mathbf{x}(T) \end{aligned} \tag{1.40}$$

where $J \in [0, T]$, $\tau_k \in (0, T)(k = 1, \cdots, q)$ and $\Psi_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \cdots, q$.

Definition 1.7.7. *The function $\boldsymbol{v} \in \mathcal{PC}^1[J, \mathbb{R}^n]$ is said to be a lower solution of system (1.40) if*

$$\begin{aligned} \dot{\boldsymbol{v}}(t) &\preceq \boldsymbol{f}(t, \boldsymbol{v}(t)), \quad t \neq \tau_k, \quad t \in J, \\ \boldsymbol{v}(\tau_k^+) &\preceq \Psi_k(\boldsymbol{v}(\tau_k)), \quad k = 1, \cdots, q, \\ \boldsymbol{v}(0) &\preceq \boldsymbol{v}(T) \end{aligned} \tag{1.41}$$

and $\boldsymbol{v}$ is not a solution of system (1.40). ⊠

Definition 1.7.8. *The function $\boldsymbol{v} \in \mathcal{PC}^1[J, \mathbb{R}^n]$ is said to be an upper solution of system (1.40) if*

$$\begin{aligned} \dot{\boldsymbol{v}} &\succeq \boldsymbol{f}(t, \boldsymbol{v}), \quad t \neq \tau_k, \quad t \in J. \\ \boldsymbol{v}(\tau_k^+) &\succeq \Psi_k(\boldsymbol{v}(\tau_k)), \quad k = 1, \cdots, q. \\ \boldsymbol{v}(0) &\succeq \boldsymbol{v}(T) \end{aligned} \tag{1.42}$$

and $\boldsymbol{v}$ is not a solution of system (1.40). ⊠

Remark 1.7.1. In Definitions 1.7.7 and 1.7.8, that $\boldsymbol{v}$ is not a solution of system (1.40) means that at least one inequality should be satisfied. ♦

Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ then we have the following two generalized derivatives of a Lyapunov function $V(t, \boldsymbol{x})$ as follows:

$$\begin{aligned} D^+V(t, \boldsymbol{x}) &\triangleq \lim_{\delta \to 0^+} \sup \frac{V(t+\delta, \boldsymbol{x} + \delta \boldsymbol{f}(t, \boldsymbol{x})) - V(t, \boldsymbol{x})}{\delta}, \\ D_-V(t, \boldsymbol{x}) &\triangleq \lim_{\delta \to 0^-} \inf \frac{V(t+\delta, \boldsymbol{x} + \delta \boldsymbol{f}(t, \boldsymbol{x})) - V(t, \boldsymbol{x})}{\delta}. \end{aligned}$$

Note that if $V \in C^1[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, then we have

$$D^+V(t, \boldsymbol{x}) = D_-V(t, \boldsymbol{x}) = \frac{\partial V(t, \boldsymbol{x})}{\partial t} + \frac{\partial V(t, \boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}).$$

Definition 1.7.9. *Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. Then $\lambda(t)$ is integrally positive if*

$$\int_{\wp} \lambda(s) ds = \infty \quad \textit{whenever} \quad \wp = \bigcup_{i=1}^{\infty} [a_i, b_i],$$
$$a_i < b_i < a_{i+1}, \quad \textit{and} \quad b_i - a_i \geq \delta > 0.$$

⊠

The following lemma gives one important impulsive integral inequality:

Lemma 1.7.1. *For $t \geq t_0$ let a nonnegative piecewise continuous function $u(t)$ satisfy*

$$u(t) \leq c + \int_{t_0}^{t} v(s) u(s) ds + \sum_{t_0 \leq \tau_i < t} b_i u(\tau_i) \tag{1.43}$$

where $c \geq 0$, $b_i \geq 0$, $v(s) > 0$. $u(t)$ has discontinuous points of the first kind at τ_i. Then we have

$$u(t) \leq c \prod_{t_0 \leq \tau_i < t} (1 + b_i) \exp\left\{ \int_{t_0}^{t} v(s) ds \right\}. \tag{1.44}$$

⊠

2. Linear Impulsive Control

In a linear impulsive control system, the plant and the impulsive control laws are both linear.

2.1 Linear Impulsive Control System with Constant Parameters

If parameters in the plant are constant, then a linear impulsive control system with fixed-time impulses is given by

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x}, & t \neq \tau_k, \\ \boldsymbol{y} = C\boldsymbol{x}, & \\ \Delta\boldsymbol{x} = P\boldsymbol{y} = PC\boldsymbol{x}, & t = \tau_k, \quad k \in \mathbb{N} \end{cases} \qquad (2.1)}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{y} \in \mathbb{R}^m$ is the output, $C \in \mathbb{R}^{m\times n}$ and $P \in \mathbb{R}^{n\times m}$. Observe that the impulsive feedback law is linear. We assume that

$$t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots, \quad \lim_{k\to\infty} \tau_k = \infty.$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be any a solution of (2.1) with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, then we have

$$\begin{aligned} \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 e^{A(t-\tau_k)} \prod_{j=1}^{k} (I + PC) e^{A(\tau_j - \tau_{j-1})}, \\ \tau_0 = t_0, \quad t \in (\tau_k, \tau_{k+1}]. \end{aligned} \qquad (2.2)$$

Since the impulsive control effects at moments $\{\tau_k\}$, the system (2.1) is a nonautonomous system. And the solutions of (2.1) is no longer invariant with respect to shifts. Given arbitrary matrices A and PC, it is difficult to say something on the property of the solutions of (2.1). Therefore even a linear impulsive control system is more difficult to study than the corresponding

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 17−33, 2001.

continuous control systems. However, if the matrices A and PC are commute then we can get a simpler form of the solution in (2.2) as

$$\boldsymbol{x}(t, \boldsymbol{x}_0) = e^{A(t-t_0)}(I + PC)^{\mathfrak{N}(t_0,t)}\boldsymbol{x}_0 \tag{2.3}$$

where $\mathfrak{N}(t_0, t)$ denotes the number of impulses in time interval $[t_0, t)$. From (2.3) it follows that the property of solutions depends on the eigenvalues of the matrices A and PC and the $\{\tau_k\}$ sequence. If we use an equidistant impulse moment generating law, i.e., $\tau_k = \tau_1 + (k-1)\delta$, where $\delta > 0$ is the impulse interval, and assume that $(I + PC)$ is nonsingular, then the solutions in (2.3) can be further simplified as

$$\begin{aligned}\boldsymbol{x}(t, \boldsymbol{x}_0) = {} & e^{A(\tau_1-t_0)}(I + PC)\exp\{-\alpha \ln(I + PC)\} \\ & \times \exp\left\{[A + \frac{1}{\delta}\ln(I + PC)](t - \tau_1)\right\}\boldsymbol{x}_0\end{aligned} \tag{2.4}$$

where

$$\alpha = \frac{t-\tau_1}{\delta} - \left\lfloor \frac{t-\tau_1}{\delta} \right\rfloor$$

and $\lfloor x \rfloor$ denotes the integer part of $x \geq 0$. Then we have the following theorem.

Theorem 2.1.1. *Assume A and PC are commute, $\{\tau_k\}$ is equidistant with interval δ, $0 < \delta < \infty$, then the trivial solution of the impulsive control system (2.1) is*

1. *asymptotically stable if all eigenvalues of the matrix $[A + \frac{1}{\delta}\ln(I + PC)]$ are in the left half s-plane;*
2. *unstable if at least one eigenvalue of the matrix $[A + \frac{1}{\delta}\ln(I + PC)]$ is in the right half s-plane.*

⊠

Remark 2.1.1. For the corresponding conventional linear control system

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} \\ \boldsymbol{u} = P\boldsymbol{y} \end{cases} \tag{2.5}$$

the property of solutions are determined solely by matrix $(A + PC)$. From Theorem 2.1.1 it follows that for a linear impulsive system the property of solutions is determined by A, PC and δ. On the one hand, if the plant is unstable, by choosing proper PC and δ, we can stabilize it "impulsively". On the other hand, we can also distabilize a stable plant "impulsively". To make some stable plants unstable by impulsive control can find applications to chaos generator design and this mechanism maybe used by real neurons to generate complex behaviors for the purpose of information processing. ♦

If A and PC are not commute then we have the following theorem:

Theorem 2.1.2. *Assume that $\{\tau_k\}$ is equidistant with impulse interval δ, $0 < \delta < \infty$, then the trivial solution of the impulsive control system (2.1) is asymptotically stable if and only if all eigenvalues of the matrix $(I+PC)e^{A\delta}$, λ_i, $i = 1, 2, \cdots, n$, satisfy $|\lambda_i| < 1$.* ⊠

Proof. From the assumptions it follows that the spectral radius of $(I + PC)e^{A\delta}$, $\rho((I + PC)e^{A\delta})$, satisfies $\rho((I + PC)e^{A\delta}) < 1$. From Theorem 5.3.4 on page 198 of [24] it follows that

$$\lim_{k\to\infty} [(I + PC)e^{A\delta}]^k = 0.$$

From (2.2) if follows that

$$\begin{aligned} \boldsymbol{x}(t, \boldsymbol{x}_0) &= \boldsymbol{x}_0 e^{A(t-\tau_k)} \prod_{j=1}^{k} (I + PC)e^{A(\tau_j - \tau_{j-1})}, \\ &= \boldsymbol{x}_0 e^{A(t-\tau_k)} [(I + PC)e^{A\delta}]^k, \\ &\quad \tau_0 = t_0, \quad t \in (\tau_k, \tau_{k+1}], \end{aligned} \tag{2.6}$$

from which we know

$$\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{x}_0) = 0 \tag{2.7}$$

for any $\|\boldsymbol{x}_0\| < \infty$. ■

If $\{\tau_k\}$ is not equidistant, then we have the following theorem.

Theorem 2.1.3. *If*

$$0 < \alpha = \inf_{k\in\mathbb{N}} (\tau_{k+1} - \tau_k) \le \sup_{k\in\mathbb{N}} (\tau_{k+1} - \tau_k) = \beta < \infty$$

then the trivial solution of the impulsive control system (2.1) is asymptotically stable if $\|e^{A(\tau_k - \tau_{k-1})}(I + PC)\| < 1$, $k \in \mathbb{N}$. ⊠

Proof. From (2.2) it follows that

$$\begin{aligned} \boldsymbol{x}(t, \boldsymbol{x}_0) &= \boldsymbol{x}_0 e^{A(t-\tau_k)} \prod_{t_0<\tau_j<t} (I + PC)e^{A(\tau_j - \tau_{j-1})}, \\ &\quad \tau_0 = t_0, \quad t \in (\tau_k, \tau_{k+1}] \end{aligned} \tag{2.8}$$

from which we have

$$\begin{aligned} \|\boldsymbol{x}(t, \boldsymbol{x}_0)\| &\le \boldsymbol{x}_0 \|e^{A(t-\tau_k)}\| \prod_{t_0<\tau_j<t} \|(I + PC)e^{A(\tau_j - \tau_{j-1})}\| \\ &\quad \tau_0 = t_0, \quad t \in (\tau_k, \tau_{k+1}]. \end{aligned} \tag{2.9}$$

Since

$$0 < \alpha = \inf_{k\in\mathbb{N}}(\tau_{k+1} - \tau_k) \leq \sup_{k\in\mathbb{N}}(\tau_{k+1} - \tau_k) = \beta < \infty$$

we know that $\|e^{A(t-\tau_k)}\| < \infty$, therefore we know if $\|e^{A(\tau_k-\tau_{k-1})}(I+PC)\| < 1$, $k \in \mathbb{N}$ then

$$\lim_{t\to\infty} \|\boldsymbol{x}(t, \boldsymbol{x}_0)\| = \lim_{k\to\infty} \|\boldsymbol{x}(t, \boldsymbol{x}_0)\| = 0$$

for any $\boldsymbol{x}_0 \in \mathbb{R}^n$. ■

Theorem 2.1.4. *If A and PC are commute and*

$$0 < \theta_1 = \inf_{k\in\mathbb{N}}(\tau_{k+1} - \tau_k) \leq \sup_{k\in\mathbb{N}}(\tau_{k+1} - \tau_k) = \theta_2 < \infty, \quad k \in \mathbb{N},$$

let

$$\alpha = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A)$$

and

$$\theta_0 = \begin{cases} \theta_1, \text{ if } \alpha \geq 0 \\ \theta_2, \text{ if } \alpha < 0 \end{cases}$$

and if

$$\alpha + \frac{1}{\theta_0} \ln \max_{j=1}^{n} |1 + \lambda_j(PC)| < 0 \tag{2.10}$$

then the trivial solution of the impulsive control system (2.1) is asymptotically stable. ⊠

Proof. Since A and PC are commute, the state transition matrix of system (2.1) is given by

$$\Psi(t, t_0) = e^{A(t-t_0)}(I + PC)^{\mathfrak{N}(t_0,t)} \tag{2.11}$$

where $\mathfrak{N}(t_0, t)$ is the number of control impulses in $[t_0, t)$. Let

$$\beta = \max_{j=1}^{n} |1 + \mathrm{Re}\lambda_j(PC)|$$

then for $t > t_0$ there exist a constant $K_1 \geq 1$ and an arbitrary small $\epsilon > 0$ such that

$$\|e^{A(t-t_0)}\| \leq K_1 e^{(\alpha+\epsilon)(t-t_0)},$$

and there exists $K_2 = K_2(\epsilon) \geq 1$ such that

$$\|(I + PC)^{\mathfrak{N}(t_0,t)}\| \leq K_2(\beta + \epsilon)^{\mathfrak{N}(t_0,t)}.$$

We then have from (2.11) that

$$\begin{aligned} \|\Psi(t, t_0)\| &\leq K_1 K_2 e^{(\alpha+\epsilon)(t-t_0)}(\beta + \epsilon)^{\mathfrak{N}(t_0,t)} \\ &\leq K_1 K_2 e^{(\alpha+\epsilon)\theta_0}\left[e^{(\alpha+\epsilon)\theta_0}(\beta + \epsilon)\right]^{\mathfrak{N}(t_0,t)} \end{aligned} \tag{2.12}$$

from which we know that if

$$e^{\alpha\theta_0}\beta < 1 \qquad \Leftarrow (2.10)$$

then

$$\|\Psi(t,t_0)\| \to 0 \text{ as } t \to \infty.$$

It is because we can choose $\epsilon > 0$ small enough such that

$$e^{\alpha\theta_0}\beta < 1 \Rightarrow \qquad e^{(\alpha+\epsilon)\theta_0}(\beta+\epsilon) < 1.$$

■

Theorem 2.1.5. *Assume that*

1. *A is in real canonical form and*
$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A);$$
2. *let $\mathfrak{N}(t,t+T)$ be the number of control impulses in interval $[t,t+T)$ and the limit*
$$\lim_{T\to\infty} \frac{\mathfrak{N}(t,t+T)}{T} = p,$$
exists and is uniform with respect to $t > t_0$;
3. *let*
$$\alpha^2 = \max_{j=1}^{n} \lambda_j[(I+PC)^\top(I+PC)]$$
with $\alpha > 0$ and $\gamma + p\ln\alpha < 0$.

Then the impulsive control system (2.1) is asymptotically stable. ⊠

Proof. For any two solutions of system (2.1) $\boldsymbol{x}(t,\boldsymbol{x}_0)$ and $\boldsymbol{x}(t,\boldsymbol{y}_0)$, we have

$$\boldsymbol{x}(t,\boldsymbol{x}_0) - \boldsymbol{x}(t,\boldsymbol{y}_0) = e^{A(t-\tau_i)} \prod_{t_0<\tau_i<t} (I+PC)e^{A(\tau_j-\tau_{j-1})}(\boldsymbol{x}_0 - \boldsymbol{y}_0). \quad (2.13)$$

Since A is in real canonical form, we have

$$\|e^{At}\boldsymbol{x}\|_2 \le e^{(\gamma+\epsilon)t}\|\boldsymbol{x}\|_2$$

with $\epsilon > 0$. Then from (2.13) it follows

$$\|\boldsymbol{x}(t,\boldsymbol{x}_0) - \boldsymbol{x}(t,\boldsymbol{y}_0)\|_2 \le e^{(\gamma+\epsilon)(t-t_0)}\alpha^{n(t,t_0)}\|\boldsymbol{x}_0 - \boldsymbol{y}_0\|_2, \quad t > t_0. \quad (2.14)$$

From assumption 2 it follows that for any $\xi > 0$ there exists $K = K(\xi)$ such that

$$\alpha^{\mathfrak{N}(t_0,t)} \le Ke^{(p\ln\alpha+\xi)(t-t_0)}$$

and therefore we have

$$\|\boldsymbol{x}(t,\boldsymbol{x}_0) - \boldsymbol{x}(t,\boldsymbol{y}_0)\|_2 \le Ke^{(\gamma+p\ln\alpha+\xi+\epsilon)(t-t_0)}\|\boldsymbol{x}_0 - \boldsymbol{y}_0\|_2, \quad t > t_0.$$

Since ϵ and ξ can be chosen arbitrarily small, then from assumption 3 we finish the proof. ■

Theorem 2.1.6. *Assume that*

1. *A and PC are commute and $I + PC$ is nonsingular;*
2. *let $\mathfrak{N}(t, t+T)$ be the number of control impulses in interval $[t, t+T)$ and the limit*
$$\lim_{T\to\infty} \frac{\mathfrak{N}(t, t+T)}{T} = p,$$
exists and is uniform with respect to $t > t_0$;
3. *all eigenvalues of $A + p\ln(I + PC)$ are in the left half s-plane.*

Then the impulsive control system (2.1) is asymptotically stable. ⊠

Proof. Since A and PC are commute, for any two solutions of system (2.1) $\boldsymbol{x}(t, \boldsymbol{x}_0)$ and $\boldsymbol{x}(t, \boldsymbol{y}_0)$, we have

$$\boldsymbol{x}(t, \boldsymbol{x}_0) - \boldsymbol{x}(t, \boldsymbol{y}_0) = e^{A(t-t_0)}(I + PC)^{\mathfrak{N}(t_0,t)}(\boldsymbol{x}_0 - \boldsymbol{y}_0), \tag{2.15}$$

or

$$\begin{aligned}\boldsymbol{x}(t, \boldsymbol{x}_0) - \boldsymbol{x}(t, \boldsymbol{y}_0) = {} & e^{[A+p\ln(I+PC)](t-t_0)} \\ & \times (I + PC)^{[\mathfrak{N}(t_0,t)-(t-t_0)p]}(\boldsymbol{x}_0 - \boldsymbol{y}_0).\end{aligned} \tag{2.16}$$

From assumption 2 it follows that there are a constant $K_1 > 0$ and any a small constant $\epsilon > 0$ such that

$$\|(I + PC)^{[\mathfrak{N}(t_0,t)-(t-t_0)p]}\| \le K_1 e^{\epsilon(t-t_0)}, \quad t \ge t_0. \tag{2.17}$$

From assumption 3 it follows that there are constants $K_2 > 0$ and $\gamma > 0$ such that

$$\|e^{[A+p\ln(I+PC)]t}\| \le K_2 e^{-\gamma t}, \qquad t \ge t_0. \tag{2.18}$$

Therefore, from (2.16) it follows that

$$\|\boldsymbol{x}(t, \boldsymbol{x}_0) - \boldsymbol{x}(t, \boldsymbol{y}_0)\| \le K_1 K_2 e^{-(\gamma-\epsilon)(t-t_0)}\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| \quad t \ge t_0. \tag{2.19}$$

Since ϵ can be chosen arbitrarily small such that $\gamma - \epsilon < 0$, we finish the proof. ■

2.2 Impulsive Control of Time-varying Linear Systems

Let us consider the following time-varying linear impulsive control system:

$$\boxed{\left\{\begin{array}{l}\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x},\ t \ne \tau_k \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x}, \quad t = \tau_k, \quad k = 1, 2, \cdots,\end{array}\right. \tag{2.20}}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is continuous matrix bounded for $t \geq t_0$, $B_k \in \mathbb{R}^{n\times n}$ are matrices uniformly bounded with respect to $k \in \mathbb{N}$ and

$$t_0 < \tau_1 < \tau_2 < \cdots, \qquad \lim_{k\to\infty} \tau_k = \infty.$$

Theorem 2.2.1. *Assume that*

1.
$$\inf_{i\in\mathbb{N}} |\det(I + B_i)| \geq \delta > 0; \tag{2.21}$$

2. *let* $\mathfrak{N}(t_0, t)$ *be the number of control impulses in the interval* $[t_0, t)$ *and*
$$\lim_{t\to\infty} \frac{\mathfrak{N}(t_0, t)}{t} = p \tag{2.22}$$
exists and is finite;
3. *the largest eigenvalue of* $\frac{1}{2}[A(t) + A^\top(t)]$, λ_n, *satisfies*
$$\lambda_n \leq \gamma \tag{2.23}$$
for all $t \geq t_0$*;*
4. *let* Λ_i *be the largest eigenvalues of matrices* $(I + B_i^\top)(I + B_i)$ *such that*
$$\Lambda_i^2 \leq \alpha, \qquad i \in \mathbb{N}; \tag{2.24}$$

5.
$$\gamma + p \ln \sqrt{\alpha} < 0. \tag{2.25}$$

Then the control system (2.20) is asymptotically stable. ⊠

Proof. For any $\boldsymbol{x}_0 \in \mathbb{R}^n$ we have

$$\begin{aligned} \overline{\lim_{t\to\infty}} \frac{\ln \|\boldsymbol{x}(t, \boldsymbol{x}_0)\|}{t} &\leq \overline{\lim_{t\to\infty}} \frac{1}{t} \left[\int_{t_0}^{t} \lambda_n(s) ds + \sum_{t_0 < \tau_i < t} \ln \Lambda_i \right] \\ &\leq \gamma + p \ln \sqrt{\alpha} < 0. \end{aligned} \tag{2.26}$$

The proof is completed. ■

Let us then consider the following impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + P(t)\boldsymbol{x}, \ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x} + J_k\boldsymbol{x}, \ t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \qquad (2.27)}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $P(t) \in \mathbb{R}^{n\times n}$ is a continuous or piecewise continuous matrix for $t \geq t_0$, $J_k \in \mathbb{R}^{n\times n}$ are constant matrices, $k \in \mathbb{N}$. Let us also consider the following reference system

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x}, & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x}, t = \tau_k, & k = 1, 2, \cdots. \end{cases} \tag{2.28}$$

We then have the following theorem.

Theorem 2.2.2. *Assume that*

1. *the solutions of the reference system (2.28) are exponentially stable;*
2. *for sufficiently large t and $k \in \mathbb{N}$, we have*

$$\|P(t)\| < \xi, \qquad \|J_k\| < \xi \tag{2.29}$$

   where $\xi > 0$ is sufficiently small;

3. $$0 < \theta_1 \leq \tau_{k+1} - \tau_k \leq \theta_2, \qquad k \in \mathbb{N}.$$

Then the solutions of impulsive control system (2.27) are exponentially stable. ⊠

Proof. Let $\Psi(t,s)$ be the state transition matrix of system (2.28), then from assumption 1 it follows that there are $K > 0$ and $\gamma > 0$ such that

$$\|\Psi(t,s)\| \leq Ke^{-\gamma(t-s)}, \qquad t \geq s. \tag{2.30}$$

For any two solutions of system (2.27), $\boldsymbol{x}(t,\boldsymbol{x}_0)$ and $\boldsymbol{x}(t,\boldsymbol{y}_0)$, we have

$$\begin{aligned} \boldsymbol{x}(t,\boldsymbol{x}_0) - \boldsymbol{x}(t,\boldsymbol{y}_0) = {} & \Psi(t,t_0)(\boldsymbol{x}_0 - \boldsymbol{y}_0) \\ & + \int_{t_0}^{t} \Psi(t,s)P(s)[\boldsymbol{x}(s,\boldsymbol{x}_0) - \boldsymbol{x}(s,\boldsymbol{y}_0)]ds \\ & + \sum_{t_0<\tau_k<t} \Psi(t,\tau_k)J_k[\boldsymbol{x}(\tau_k,\boldsymbol{x}_0) - \boldsymbol{x}(\tau_k,\boldsymbol{y}_0)] \end{aligned}$$

from which and (2.30) it follows

$$\begin{aligned} & e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0) - \boldsymbol{x}(t,\boldsymbol{y}_0)\| \leq \\ & K\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| + \int_{t_0}^{t} Ke^{\gamma(s-t_0)}\|P(s)\|\|\boldsymbol{x}(s,\boldsymbol{x}_0) - \boldsymbol{x}(s,\boldsymbol{y}_0)\|ds \\ & + \sum_{t_0<\tau_k<t} Ke^{\gamma(\tau_k-t_0)}\|J_k\|\|\boldsymbol{x}(\tau_k,\boldsymbol{x}_0) - \boldsymbol{x}(\tau_k,\boldsymbol{y}_0)\|. \end{aligned}$$

Let us choose $T > t_0$ such that for any $t > T$ and $\tau_k > T$, assumption 2 holds, then we have

$$\begin{aligned}
&e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)-\boldsymbol{x}(t,\boldsymbol{y}_0)\| \leq \\
&\phi + \int_T^t K\xi e^{\gamma(s-t_0)}\|\boldsymbol{x}(s,\boldsymbol{x}_0)-\boldsymbol{x}(s,\boldsymbol{y}_0)\|ds \\
&+ \sum_{T\leq\tau_k<t} K\xi e^{\gamma(\tau_k-t_0)}\|\boldsymbol{x}(\tau_k,\boldsymbol{x}_0)-\boldsymbol{x}(\tau_k,\boldsymbol{y}_0)\|,
\end{aligned}$$

where

$$\begin{aligned}
\phi = K\|\boldsymbol{x}_0-\boldsymbol{y}_0\| &+ \int_{t_0}^T Ke^{\gamma(s-t_0)}\|P(s)\|\|\boldsymbol{x}(s,\boldsymbol{x}_0)-\boldsymbol{x}(s,\boldsymbol{y}_0)\|ds \\
&+ \sum_{t_0<\tau_k<T} Ke^{\gamma(\tau_k-t_0)}\|J_k\|\|\boldsymbol{x}(\tau_k,\boldsymbol{x}_0)-\boldsymbol{x}(\tau_k,\boldsymbol{y}_0)\|. \qquad (2.31)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)-\boldsymbol{x}(t,\boldsymbol{y}_0)\| \leq \\
&\phi + \int_{t_0}^t K\xi e^{\gamma(s-t_0)}\|\boldsymbol{x}(s,\boldsymbol{x}_0)-\boldsymbol{x}(s,\boldsymbol{y}_0)\|ds \\
&+ \sum_{t_0<\tau_k<t} K\xi e^{\gamma(\tau_k-t_0)}\|\boldsymbol{x}(\tau_k,\boldsymbol{x}_0)-\boldsymbol{x}(\tau_k,\boldsymbol{y}_0)\|. \qquad (2.32)
\end{aligned}$$

Then from Lemma 1.7.1 we have

$$e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)-\boldsymbol{x}(t,\boldsymbol{y}_0)\| \leq \phi(1+K\xi)^{\mathfrak{N}(t_0,t)}e^{K\xi(t-t_0)}, \qquad (2.33)$$

and from assumption 3 we have

$$\begin{aligned}
&\|\boldsymbol{x}(t,\boldsymbol{x}_0)-\boldsymbol{x}(t,\boldsymbol{y}_0)\| \leq \\
&\phi\exp\left\{-\left(\gamma-K\xi-\frac{1}{\theta_1}\ln(1+K\xi)\right)(t-t_0)\right\}. \qquad (2.34)
\end{aligned}$$

If ξ is so small that

$$\gamma - K\xi - \frac{1}{\theta_1}\ln(1+K\xi) > 0, \qquad (2.35)$$

then we have

$$\|\boldsymbol{x}(t,\boldsymbol{x}_0)-\boldsymbol{x}(t,\boldsymbol{y}_0)\| \to 0 \text{ as } t\to\infty.$$

■

Similar to Theorem 2.1.5 we have the following theorem.

Theorem 2.2.3. *Assume that*

1. *A is in real canonical form and let*

$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A);$$

2. *let* $\mathfrak{N}(t, t+T)$ *be the number of control impulses in interval* $[t, t+T)$ *and the limit*
$$\lim_{T\to\infty} \frac{\mathfrak{N}(t, t+T)}{T} = p$$
exists and is uniform with respect to $t > t_0$*;*
3. *let*
$$\alpha^2 = \max_{j=1}^{n} \lambda_j[(I+B)^\top (I+B)]$$
with $\alpha > 0$ *and* $\gamma + p \ln \alpha < 0$*;*
4. *for sufficiently large* t *and* $k \in \mathbb{N}$*, we have*
$$\|P(t)\| < \xi, \qquad \|J_k\| < \xi \tag{2.36}$$
where $\xi > 0$ *is sufficiently small.*

Then the solutions of impulsive control system (2.27) are asymptotically stable. ⊠

Similarly, we have the following theorem.

Theorem 2.2.4. *Assume that*

1. A *is in real canonical form and let*
$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A);$$
2. $\tau_k, k \in \mathbb{N}$ *satisfy* $0 < \theta_1 \le \tau_{k+1} - \tau_k \le \theta_2$;
3. *let*
$$\alpha^2 = \max_{j=1}^{n} \lambda_j[(I+B)^\top (I+B)]$$
with $\alpha > 0$ *and*
$$\gamma + \frac{1}{\theta} \ln \alpha < 0$$
where
$$\theta = \begin{cases} \theta_1, & \text{if } \alpha \ge 1, \\ \theta_2, & \text{if } 0 < \alpha < 1; \end{cases}$$
4. *for sufficiently large* t *and* $k \in \mathbb{N}$*, we have*
$$\|P(t)\| < \xi, \qquad \|J_k\| < \xi \tag{2.37}$$
where $\xi > 0$ *is sufficiently small.*

Then the solutions of system (2.27) are asymptotically stable. ⊠

2.3 Controllability of Linear Impulsive Control Systems

In this section we study the controllability of type-I and type-II linear impulsive control systems.

2.3.1 Time-varying Cases

In this section we consider the impulsive control of the following time-varying linear system:

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x} \tag{2.38}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $A \in C[\mathbb{R}_+, \mathbb{R}^{n\times n}]$. We denote the fundamental matrix of system (2.38) as $\Phi(t)$. The corresponding impulsive controlled system is given by

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x} + B(t)\boldsymbol{u}(t), & t \neq \tau_k \\ \Delta\boldsymbol{x} = B_k\boldsymbol{u}_k, & t = \tau_k, \quad k \in \mathbb{N} \end{cases} \tag{2.39}$$

where $\boldsymbol{u}(t) \in \mathbb{R}^{\tilde{m}}$, $B \in C[\mathbb{R}_+, \mathbb{R}^{n\times\tilde{m}}]$, $\tilde{m} < n$, $\boldsymbol{u}_k \in \mathbb{R}^m$, $B_k \in \mathbb{R}^{n\times m}$ and $m < n$.

Definition 2.3.1. *The time-varying linear system (2.38) is called impulsively controllable if for all vector pair $\boldsymbol{x}_1 \in \mathbb{R}^n$, $\boldsymbol{x}_2 \in \mathbb{R}^n$, and any open time interval (t_1, t_2), there exist real numbers $\tau_k \in [t_1, t_2]$, $k = 1, 2, \cdots, p < \infty$, $\tau_1 < \tau_2 < \cdots < \tau_p$, and control vectors $\boldsymbol{u}_k \in \mathbb{R}^m$, such that the controlled system (2.39) has a solution $\boldsymbol{x}(t)$ existing on $[t_1, t_2]$ satisfying $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$ and $\boldsymbol{x}(t_2^+) = \boldsymbol{x}_2$.* ⊠

Theorem 2.3.1. *Given two terminal conditions $(t_1, \boldsymbol{x}_1)$ and $(t_2, \boldsymbol{x}_2)$, the control system in (2.39) is controllable at time t_1 if and only if*

1. *the $n \times m$ matrix function $\Phi(t)\Phi^{-1}(t_1)B(t)$ are linearly independent on $[t_1, t_2]$,*
   or,
2. $\mathrm{Rank}([\Phi^{-1}(\tau_1)B_1 \;\; \Phi^{-1}(\tau_2)B_2 \;\; \cdots \;\; \Phi^{-1}(\tau_p)B_p]) = n.$

⊠

Proof. In control system (2.39) there are two kinds of control signals generated by linear continuous control laws and linear impulsive control law. The controllability of (2.39) can be considered with respect to linear continuous control law and linear impulsive control law, respectively. Therefore the proof consists of two steps. In the first step we set $\boldsymbol{u}_k = 0$ for all $k = 0, 1, 2, \cdots$, then the control system (2.39) becomes a conventional linear control system. Then the control system in (2.39) is controllable at time t_1 if and only if the $n \times \tilde{m}$ matrix function $\Phi(t)\Phi^{-1}(t_1)B(t)$ are linearly independent on $[t_1, t_2]$. We first prove the sufficiency. If the rows of $\Phi(t)\Phi^{-1}(t_1)B(t)$ are linearly independent on $[t_1, t_2]$, then the $n \times n$ Grammian matrix

$$W(t_1, t_2) \triangleq \int_{t_1}^{t_2} \Phi(t_1, t)B(t)B^\top(t)\Phi^\top(t_1, t)dt \tag{2.40}$$

is nonsingular and thus $W^{-1}(t_1, t_2)$ exists and $\Phi(s, p) \triangleq \Phi(s)\Phi^{-1}(p)$. Since

$$\begin{aligned}
\boldsymbol{x}(t_2) &= \Phi(t_2, t_1)\left\{\boldsymbol{x}_1 - \left[\int_{t_1}^{t_2} \Phi(t_1, t)B(t)B^\top(t)\Phi^\top(t_1, t)dt\right]\right. \\
&\qquad \left.[W^{-1}(t_1, t_2)(\boldsymbol{x}_1 - \Phi(t_1, t_2)\boldsymbol{x}_2)]\right\} \\
&= \Phi(t_2, t_1)[\boldsymbol{x}_1 - W(t_1, t_2)W^{-1}(t_1, t_2)(\boldsymbol{x}_1 - \Phi(t_1, t_2)\boldsymbol{x}_2)] \\
&= \Phi(t_2, t_1)\Phi(t_1, t_2)\boldsymbol{x}_2 = \boldsymbol{x}_2
\end{aligned} \tag{2.41}$$

we know that

$$\boldsymbol{u}(t) = -B^\top(t)\Phi^\top(t_1, t)W^{-1}(t_1, t_2)(\boldsymbol{x}_1 - \Phi(t_1, t_2)\boldsymbol{x}_2) \tag{2.42}$$

is a control input that satisfying the terminal conditions $(t_1, \boldsymbol{x}_1)$ and $(t_2, \boldsymbol{x}_2)$. Thus the control system in (2.39) is controllable.

We then prove the necessity which implies that if the control system in (2.39) is controllable then the n rows of $\Phi(t)\Phi^{-1}(t_1)B(t)$ are linearly independent. Let us assume that (2.39) is controllable but the rows of $\Phi(t)\Phi^{-1}(t_1)B(t)$ are linearly dependent. Thus, there exists a nonzero and constant $1 \times n$ real vector $\boldsymbol{w}$ such that

$$\boldsymbol{w}\Phi(t_1, t)B(t) = 0 \quad \text{for all } t \text{ in } [t_1, t_2]. \tag{2.43}$$

We can choose $\boldsymbol{x}(t_1) = \boldsymbol{x}_1 = \boldsymbol{w}^\top$ as an initial state and $\boldsymbol{x}(t_2) = \boldsymbol{x}_2 = 0$ as a terminal state because the system is controllable. Then we have

$$0 = \Phi(t_2, t_1)\left[\boldsymbol{w}^\top + \int_{t_1}^{t_2} \Phi(t_1, t)B(t)\boldsymbol{u}(t)dt\right] \tag{2.44}$$

from which we have

$$0 = \boldsymbol{w}\boldsymbol{w}^\top + \int_{t_1}^{t_2} \boldsymbol{w}\Phi(t_1, t)B(t)\boldsymbol{u}(t)dt. \tag{2.45}$$

Followed (2.43), (2.45) reduces into

$$\boldsymbol{w}\boldsymbol{w}^\top = 0 \tag{2.46}$$

which in turn implies that $\boldsymbol{w} = 0$. This is a contradiction.

In the second step, we assume that $\boldsymbol{u}(t) = 0$ for $t \in [t_1, t_2]$. For given τ_k, $\boldsymbol{u}_k$, $k = 1, 2, \cdots, p$, the unique solution of impulsive control system (2.39) satisfying the initial condition $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$ is given by

$$\begin{cases} \boldsymbol{x}(t) = \Phi(t)\Phi^{-1}(t_1)\boldsymbol{x}_1, & t \in (t_1, \tau_1], \\ \boldsymbol{x}(t) = \Phi(t)\Phi^{-1}(\tau_k)\boldsymbol{x}(\tau_k^+), & t \in (\tau_k, \tau_{k+1}], \quad k = 1, 2, \cdots, p. \end{cases} \tag{2.47}$$

From (2.39) and (2.47) it follows

$$\boldsymbol{x}(t_2^+) = \Phi(t_2)\left(\Phi^{-1}(t_1)\boldsymbol{x}_1 + \sum_{k=1}^{p}\Phi^{-1}(\tau_k)B_k\boldsymbol{u}_k\right). \tag{2.48}$$

From the definition of impulsive controllability we know that $\boldsymbol{x}(t_2^+) = \boldsymbol{x}_2$ should be satisfied, we then have

$$\sum_{k=1}^{p}\Phi^{-1}(\tau_k)B_k\boldsymbol{u}_k = \Phi^{-1}(t_2)\boldsymbol{x}_2 - \Phi^{-1}(t_1)\boldsymbol{x}_1. \tag{2.49}$$

Let us define

$$\begin{aligned}\boldsymbol{v} &\triangleq \Phi^{-1}(t_2)\boldsymbol{x}_2 - \Phi^{-1}(t_1)\boldsymbol{x}_1,\\ W &\triangleq [\Phi^{-1}(\tau_1)B_1 \;\; \Phi^{-1}(\tau_2)B_2 \;\; \cdots \;\; \Phi^{-1}(\tau_p)B_p],\\ U^\top &\triangleq (\boldsymbol{u}_1^\top, \boldsymbol{u}_2^\top, \cdots, \boldsymbol{u}_p^\top)\end{aligned} \tag{2.50}$$

then (2.49) can be rewritten as

$$WU = \boldsymbol{v}.$$

Because $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and therefore $\boldsymbol{v}$ are arbitrary, the impulsive controllability of system (2.38) is equivalent to

$$\mathrm{Rank}(W) = n. \tag{2.51}$$

■

If the condition in (2.51) is not satisfied, then we can only find *partial impulsive controllability* of impulsive control systems. Then our question becomes as follows: Given two pairs of terminal conditions $(t_1, \boldsymbol{x}_1)$ and $(t_2, \boldsymbol{x}_2)$ under which conditions we can find a solution of system (2.39) to satisfy both of them. From (2.49) it follows that the necessary and sufficient condition for the existence of such a solution is given by

$$\boldsymbol{v} \in \mathrm{col}\{W\}, \tag{2.52}$$

where $\mathrm{col}\{W\}$ is the column space of W.

2.3.2 Time-invariant Cases

If the $A(t)$ matrix is constant and there is no continuous control law, then the linear impulsive control system (2.39) can be recast into

$$\boxed{\begin{cases}\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}, \quad t \neq \tau_k\\ \Delta\boldsymbol{x} = B_k\boldsymbol{u}_k,\, t = \tau_k, \quad k = 1, 2, \cdots.\end{cases} \qquad (2.53)}$$

In this case, the fundamental matrix is given by

$$\Phi(t) = e^{At}.$$

Followed (2.50) we have

$$W = [e^{-A\tau_1}B_1 \quad e^{-A\tau_2}B_2 \quad \cdots \quad e^{-A\tau_p}B_p]. \tag{2.54}$$

By using Cayley-Hamilton theorem we can use an nth-order polynomial in A to represent e^{At}. Let

$$\Lambda(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i$$

be the characteristic polynomial of A and let the scalar functions $f_i(\lambda)$, $i = 0, 1, \cdots, n-1$, be the unique solution to the following system

$$\begin{pmatrix} \frac{df_0(\lambda)}{d\lambda} \\ \frac{df_1(\lambda)}{d\lambda} \\ \frac{df_2(\lambda)}{d\lambda} \\ \vdots \\ \frac{df_{n-1}(\lambda)}{d\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \begin{pmatrix} f_0(\lambda) \\ f_1(\lambda) \\ f_2(\lambda) \\ \vdots \\ f_{n-1}(\lambda) \end{pmatrix}, \tag{2.55}$$

satisfying

$$\begin{aligned} &f_0(0) = 1, \\ &f_i(0) = 0, \quad i = 1, 2, \cdots, n-1. \end{aligned} \tag{2.56}$$

We then have the following lemmas.

Lemma 2.3.1.

$$e^{A\lambda} = \sum_{i=0}^{n-1} f_i(\lambda) A^i.$$

⊠

Proof.

$$\begin{aligned} \frac{d}{d\lambda} \sum_{i=0}^{n-1} f_i(\lambda) A^i = {}& f_0(\lambda)A + f_1(\lambda)A^2 + \cdots + f_{n-2}(\lambda)A^{n-1} \\ &+ f_{n-1}(\lambda)(-a_0 I - a_1 A - \cdots - a_{n-1}A^{n-1}). \end{aligned} \tag{2.57}$$

By Cayley-Hamilton theorem we know that

$$A^n = -a_0 I - a_1 A - \cdots - a_{n-1} A^{n-1}.$$

We then have

$$\frac{d}{d\lambda}\sum_{i=0}^{n-1} f_i(\lambda)A^i = A\sum_{i=0}^{n-1} f_i(\lambda)A^i. \tag{2.58}$$

Also,

$$\left.\sum_{i=0}^{n-1} f_i(\lambda)A^i\right|_{\lambda=0} = \sum_{i=0}^{n-1} f_i(0)A^i = A^0 = I. \tag{2.59}$$

Thus, the solution to (2.58) and (2.59) is unique and $e^{A\lambda}$ satisfies (2.58) and (2.59). ■

Lemma 2.3.2. *The functions $f_i(\lambda)$, $i = 0, 1, \cdots, n-1$, defined in (2.55) and (2.56) are linearly independent in any open interval (t_1, t_2).* ⊠

Proof. Suppose

$$\sum_{i=0}^{n-1} c_i f_i(\lambda) = 0$$

for some constants c_i, it follows from (2.56) that $c_0 = 0$ and we also have

$$\frac{d}{d\lambda}\sum_{i=0}^{n-1} c_i f_i(\lambda) = 0.$$

Then followed (2.55) we have

$$\sum_{i=1}^{n-1} c_i[f_{i-1}(\lambda) - a_i f_{n-1}(\lambda)] = 0. \tag{2.60}$$

If we define $f_i(\lambda) = 0$ for $i < 0$, then (2.60) can be rewritten as

$$\sum_{i=0}^{n-1} c_i[f_{i-1}(\lambda) - a_i f_{n-1}(\lambda)] = 0. \tag{2.61}$$

Evaluating (2.61) at $\lambda = 0$ and followed (2.56) we have $c_1 = 0$. We differentiate (2.61) and get

$$\sum_{i=0}^{n-1} c_i\{f_{i-2}(\lambda) - a_{i-1}f_{n-1}(\lambda) - a_i[f_{n-2}(\lambda) - a_{n-1}f_{n-1}(\lambda)]\} = 0, \tag{2.62}$$

which upon evaluation at $\lambda = 0$ gives $c_2 = 0$. By using the similar process, it is easy to show that $c_i = 0$, $i = 0, 1, \cdots, n-1$. Thus, the functions $f_i(\lambda)$ are linearly independent on any interval containing zero. Furthermore, since the functions $f_i(\lambda)$ satisfy a linear constant coefficient system of ODEs, it is known that they must be analytic. This implies that these functions are linearly independent in any open interval (t_1, t_2). ■

Lemma 2.3.3. *If $f_1(\lambda)$, $f_2(\lambda)$, $\cdots$, $f_n(\lambda)$ are linearly independent functions in every open interval, then for a given $p \le n$, and any open interval (t_1, t_2), there exist real numbers τ_i, $i = 1, 2, \cdots, p$, such that $t_1 \le \tau_1 < \tau_2 < \cdots < \tau_p \le t_2$, and the matrix*

$$\Gamma_p \triangleq \begin{pmatrix} f_1(\tau_1) & f_1(\tau_2) & \cdots & f_1(\tau_p) \\ f_2(\tau_1) & f_2(\tau_2) & \cdots & f_2(\tau_p) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(\tau_1) & f_n(\tau_2) & \cdots & f_n(\tau_p) \end{pmatrix}$$

has rank p. ⊠

Proof. We can choose $\tau_1 \in [t_1, t_2)$ such that Γ_1 has rank one. Suppose that we choose time moments τ_i, $i = 1, 2, \cdots, q$, satisfying $t_1 \le \tau_1 < \tau_2 < \cdots < \tau_q \le t_2$ such that Γ_q has rank q, $q < p \le n$. For fixed k we view Γ_k as a linear transformation $T_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_k \boldsymbol{x} = (\Gamma_k \;\; \mathbf{0}_k)^\top \boldsymbol{x}$ where $\mathbf{0}_k$ is the $n \times (n-k)$ zero matrix. It follows that $\dim\{\ker\ T_q\} = n - q > 0$. Choose nonzero $\boldsymbol{w} \in \ker\ T_q$ and suppose $\boldsymbol{w} \in \ker\ T_{q+1}$ for all $\tau_{q+1} \in (\tau_q, t_2)$. It follows that

$$(\Gamma_{q+1} \;\; \mathbf{0}_{q+1})^\top \boldsymbol{w} = 0.$$

hence

$$[f_1(\tau_{q+1}), f_2(\tau_{q+1}), \cdots, f_n(\tau_{q+1})]\boldsymbol{w} = 0, \qquad \forall \tau_{q+1} \in (\tau_q, t_2). \tag{2.63}$$

However, the function $f_i(\tau)$ are linearly independent in (τ_q, t_2) so that (2.63) can not hold. This is a contradiction. Therefore, there exists a time $\tau_{q+1} \in (\tau_q, t_2)$ such that $\boldsymbol{w} \neq \ker\ T_{q+1}$, and since $\ker T_{q+1} \subset \ker\ T_q$, we have $\dim\{\ker\ T_{q+1}\} = \dim\{\ker\ T_q\} - 1 = n - (q+1)$, implying Γ_{q+1} has rank $q + 1$. Thus by mathematical induction we can construct Γ_p to have rank $p \le n$. ■

Theorem 2.3.2. *If A and $B_k \equiv B$ are constant matrices then system (2.53) is impulsively controllable if and only if*

$$\text{Rank}[B, AB, A^2B, \cdots, A^{n-1}B] = n. \tag{2.64}$$

Furthermore, the maximum number of impulses required to achieve any desired final state for a controllable constant system is $p = \lceil n/m \rceil$, where $\lceil x \rceil$ denote the smallest integer greater than or equal to x. ⊠

Proof. The controllability condition is given by (2.50) and (2.51) and followed Lemma 2.3.1 we have

$$
\begin{aligned}
W &= \left[\sum_{i=0}^{n-1} f_i(-\tau_1) A^i B, \sum_{i=0}^{n-1} f_i(-\tau_1) A^i B, \cdots, \sum_{i=0}^{n-1} f_i(-\tau_p) A^i B \right] \\
&= [B, AB, \cdots, A^{n-1}B] \\
&\quad \times \begin{bmatrix} f_0(-\tau_1)I_m & f_0(-\tau_2)I_m & \cdots & f_0(-\tau_p)I_m \\ f_1(-\tau_1)I_m & f_1(-\tau_2)I_m & \cdots & f_1(-\tau_p)I_m \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(-\tau_1)I_m & f_{n-1}(-\tau_2)I_m & \cdots & f_{n-1}(-\tau_p)I_m \end{bmatrix},
\end{aligned}
\tag{2.65}
$$

where I_m is the $m \times m$ identity matrix. We then need to show that the matrix

$$
S \triangleq \begin{bmatrix} f_0(-\tau_1)I_m & f_0(-\tau_2)I_m & \cdots & f_0(-\tau_p)I_m \\ f_1(-\tau_1)I_m & f_1(-\tau_2)I_m & \cdots & f_1(-\tau_p)I_m \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(-\tau_1)I_m & f_{n-1}(-\tau_2)I_m & \cdots & f_{n-1}(-\tau_p)I_m \end{bmatrix} \tag{2.66}
$$

has full rank that is no less than n for some choice of ordered time moments $\tau_k \in [t_1, t_2]$, $k = 1, 2, \cdots, p$, where $p = \lceil n/m \rceil$. The matrix S has full rank if and only if the matrix

$$
Q \triangleq \begin{bmatrix} f_0(-\tau_1) & f_0(-\tau_2) & \cdots & f_0(-\tau_p) \\ f_1(-\tau_1) & f_1(-\tau_2) & \cdots & f_1(-\tau_p) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(-\tau_1) & f_{n-1}(-\tau_2) & \cdots & f_{n-1}(-\tau_p) \end{bmatrix} \tag{2.67}
$$

has full rank. S has pm columns, hence $\mathrm{Rank}(S) \geq n$ implies $p \geq n/m$; this provides the lower bound for p. Let $p = \lceil n/m \rceil$, then since $m \geq 1$, $p \leq n$ and by Lemma 2.3.2 we can apply Lemma 2.3.3 so that there exist $\tau_1, \tau_2, \cdots, \tau_p$ such that Q has full rank. ■

Corollary 2.3.1. *If A and $B_k \equiv B$ are constant matrices, then the linear impulsive control system (2.53) is impulsively controllable if and only if*

$$
\mathrm{Rank}[sI - A, B] = n, \qquad \forall s \in \mathbb{C}. \tag{2.68}
$$

⊠

Proof. By using simple linear algebra we know

$$
\mathrm{Rank}[B, AB, A^2B, \cdots, A^{n-1}B] = n \tag{2.69}
$$

if and only if

$$
\mathrm{Rank}[sI - A, B] = n, \qquad \forall s \in \mathbb{C}. \tag{2.70}
$$

Then by using Theorem 2.3.2, we finish the proof. ■

Note 2.3.1. The controllability of linear impulsive system is adopted from [20]. ♦

3. Comparison Methods

In this chapter let us study the stability of impulsive control systems using comparison methods.

3.1 Single Comparison System

Let us consider the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}), & t \neq \tau_k, \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & k = 1, 2, \cdots . \end{cases} \tag{3.1}$$

Definition 3.1.1. *Comparison System*
Let $V \in \mathcal{V}_0$ and assume that

$$\begin{cases} D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x})), & t \neq \tau_k, \\ V(t, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x})), & t = \tau_k, \end{cases} \tag{3.2}$$

where $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous in $(\tau_{k-1}, \tau_k] \times \mathbb{R}$ and for each $x \in \mathbb{R}$, $k = 1, 2, \cdots$,

$$\lim_{(t,y) \to (\tau_k^+, x)} g(t, y) = g(\tau_k^+, x)$$

exists. $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing. Then the following system

$$\begin{cases} \dot{w} = g(t, w), & t \neq \tau_k, \\ w(\tau_k^+) = \psi_k(w(\tau_k)), \\ w(t_0^+) = w_0 \geq 0 \end{cases} \tag{3.3}$$

is the comparison system of (3.1). ⊠

Consider the following system

$$\begin{cases} \dot{w} = g(t, w), & t \neq \tau_k, \\ w(\tau_k^+) = \psi_k(w(\tau_k)), & \tau_k > t_0 \geq 0 \\ w(t_0) = w_0 \end{cases} \tag{3.4}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $\psi_k : \mathbb{R} \to \mathbb{R}$, we have the following definitions.

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 35−70, 2001.

Definition 3.1.2. *Let $w_{\max}(t) = w_{\max}(t, t_0, w_0)$ be a solution of (3.4) on $[t_0, t_0 + T)$. If for any solution $w(t) = w(t, t_0, w_0)$ of (3.4) on $[t_0, t_0 + T)$ the following inequality*

$$w_{\max}(t) \geq w(t), \quad t \in [t_0, t_0 + T)$$

holds, then $w_{\max}(t)$ is the maximal solution of (3.4) on $[t_0, t_0 + T)$. The minimal solution, $w_{\min}(t) = w_{\min}(t, t_0, w_0)$ on $[t_0, t_0 + T)$ satisfies the following inequality:

$$w_{\min}(t) \leq w(t), \quad t \in [t_0, t_0 + T).$$

⊠

Theorem 3.1.1. *Consider the following system*

$$\begin{cases} \dot{w} = g(t, w), & t \neq \tau_k, \\ w(\tau_k^+) = \psi_k(w(\tau_k)), & \tau_k > t_0 \geq 0 \\ w(t_0) = w_0 \end{cases} \tag{3.5}$$

Suppose that

1. *$\{\tau_k\}$ satisfies $0 \leq t_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k \to \infty} \tau_k = \infty$;*
2. *$p \in \mathcal{PC}^1[\mathbb{R}_+, \mathbb{R}]$ and $p(t)$ is left-continuous at τ_k, $k = 1, 2, \cdots$;*
3. *$g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $\psi_k : \mathbb{R} \to \mathbb{R}$, $\psi_k(w)$ is nondecreasing in w and for each $k = 1, 2, \cdots$,*

$$\begin{aligned} &D_- p(t) \leq g(t, p(t)), \quad t \neq \tau_k, \quad p(t_0) \leq w_0, \\ &p(\tau_k^+) \leq \psi_k(p(\tau_k)); \end{aligned} \tag{3.6}$$

4. *$w_{\max}(t)$ is the maximal solution of (3.5) on $[t_0, \infty)$.*

Then $p(t) \leq w_{\max}(t)$, $t \in [t_0, \infty)$. ⊠

Proof. From the classical comparison theorem we know that $p(t) \leq w_{\max}(t)$ for $t \in [t_0, \tau_1]$. Since $p(\tau_1) \leq w_{\max}(\tau_1)$ and $\psi_1(w)$ is nondecreasing we have

$$p(\tau_1^+) \leq \psi_1(p(\tau_1)) \leq \psi_1(w_{\max}(\tau_1)) = w_1^+. \tag{3.7}$$

For $t \in (\tau_1, \tau_2]$ by using classical comparison theorem we know that $p(t) \leq w_{\max}(t)$, where $w_{\max}(t) = w_{\max}(t, \tau_1, w_1^+)$ is the maximal solution of (3.5) on $t \in [\tau_1, \tau_2]$. Similarly, we have

$$p(\tau_2^+) \leq \psi_2(p(\tau_2)) \leq \psi_2(w_{\max}(\tau_2)) = w_2^+. \tag{3.8}$$

By repeating the same process we can finish the proof. ∎

Theorem 3.1.2. *Let $w_{\max}(t) = w_{\max}(t, t_0, w_0)$ be the maximal solution of (3.3) on $[t_0, \infty)$, then for any solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, $t \in [t_0, \infty)$, of (3.1), $V(t_0^+, \boldsymbol{x}_0) \leq w_0$ implies that*

$$V(t, \boldsymbol{x}(t)) \leq w_{\max}(t), \qquad t \geq t_0. \tag{3.9}$$

⊠

Proof. Let $\tilde{V}(t) = V(t, \boldsymbol{x}(t))$ for $t \neq \tau_k$ such that for a small $\delta > 0$ we have

$$\begin{aligned}\tilde{V}(t+\delta) - \tilde{V}(t) &= V(t+\delta, \boldsymbol{x}(t+\delta)) - V(t+\delta, \boldsymbol{x}(t) + \delta(\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}))) \\ &\quad + V(t+\delta, \boldsymbol{x}(t) + \delta(\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}))) - V(t, \boldsymbol{x}(t)). \quad (3.10)\end{aligned}$$

Since $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ for $t \in (\tau_k, \tau_{k+1}]$, from (3.2) we have

$$\begin{aligned}& D^+\tilde{V}(t) \leq g(t, \tilde{V}(t)), \quad t \neq \tau_k, \tilde{V}(t_0^+) \leq w_0, \\ & \tilde{V}(\tau_k^+) = V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \leq \psi_k(\tilde{V}(\tau_k)). \qquad (3.11)\end{aligned}$$

Therefore, followed Theorem 3.1.1 we finish the proof. ■

For some special case of $g(t, w)$ and $\psi_k(w)$ we have the following corollary.

Corollary 3.1.1. *If in Theorem 3.1.2 we assume that*

1. $g(t, w) = 0$, $\psi_k(w) = w$ *for all* k, *then* $V(t, \boldsymbol{x})$ *is nonincreasing in* t *and* $V(t, \boldsymbol{x}) \leq V(t_0^+, \boldsymbol{x}_0)$, $t \geq t_0$;
2. $g(t, w) = 0$, $\psi_k(w) = d_k w, d_k \geq 0$ *for all* k, *then*

$$V(t, \boldsymbol{x}) \leq V(t_0^+, \boldsymbol{x}_0) \prod_{t_0 < \tau_k < t} d_k, \qquad t \geq t_0;$$

3. $g(t, w) = -\gamma w, \gamma > 0$, $\psi_k(w) = d_k w, d_k \geq 0$ *for all* k, *then*

$$V(t, \boldsymbol{x}) \leq \left(V(t_0^+, \boldsymbol{x}_0) \prod_{t_0 < \tau_k < t} d_k \right) e^{-\gamma(t - t_0)}, \qquad t \geq t_0;$$

4. $g(t, w) = \dot{\lambda}(t) w$, $\lambda(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_k(w) = d_k w, d_k \geq 0$ *for all* k, *then*

$$V(t, \boldsymbol{x}) \leq \left(V(t_0^+, \boldsymbol{x}_0) \prod_{t_0 < \tau_k < t} d_k \right) e^{\lambda(t) - \lambda(t_0)}, \qquad t \geq t_0.$$

⊠

Then the following theorem gives sufficient conditions in a unified way for various stability criteria.

Theorem 3.1.3. *Assume that*

1. $\boldsymbol{f}(t, 0) = 0$, $\boldsymbol{u}(t, 0) = 0$, $g(t, 0) = 0$ *and* $U(k, 0) = 0$ *for all* k;
2. $V : \mathbb{R}_+ \times \mathcal{S}_\rho \to \mathbb{R}_+, \rho > 0$, $V \in \mathcal{V}_0$, $D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}))$, $t \neq \tau_k$;
3. *There exists a* $\rho_0 > 0$ *such that* $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$ *implies that* $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_\rho$ *for all* k *and* $V(t, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x}))$, $t = \tau_k$, $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$;
4. $\beta(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|)$ *on* $\mathbb{R}_+ \times \mathcal{S}_\rho$ *where* $\alpha(\cdot), \beta(\cdot) \in \mathcal{K}$.

Then the stability properties of the trivial solution of comparison system (3.3) imply the corresponding stability properties of the trivial solution of (3.1). ⊠

Proof. From condition 1 we know that the trivial solutions of both (3.1) and (3.3) exist. We then prove the equivalence of stable, uniformly stable, asymptotically stable and uniformly asymptotically stable properties of systems (3.1) and (3.3).

1. Stable: Let us suppose that the trivial solution of (3.3) is stable and let $0 < \eta < \min(\rho, \rho_0)$, then there exists an $\varepsilon_1(t_0, \eta) > 0$ such that $0 \leq w_0 < \varepsilon_1$ implies $w(t, t_0, w_0) < \beta(\eta)$ where $t \geq t_0$ and $w(t, t_0, w_0)$ is an arbitrary solution of (3.3). Let us choose $w_0 = \alpha(\|\boldsymbol{x}_0\|)$ and let $\varepsilon_2 = \varepsilon_2(\eta)$ such that $\alpha(\varepsilon_2) < \beta(\eta)$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then we have the following claim:
   Claim 3.1.3: For any solution, $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of (3.1) if $\|\boldsymbol{x}_0\| < \varepsilon$ then $\|\boldsymbol{x}(t)\| < \eta$ for $t \geq t_0$.
   If Claim 3.1.3 is not true then there are a k and a solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$ of (3.1) satisfying
   $$\eta \leq \|\boldsymbol{x}_1(t_1)\| \text{ and } \|\boldsymbol{x}_1(t)\| < \eta \text{ for } t \in [t_0, \tau_k] \tag{3.12}$$
   where $\|\boldsymbol{x}_0\| < \varepsilon$, $t_1 > t_0$ and $t_1 \in (\tau_k, \tau_{k+1}]$.
   From $0 < \eta < \min(\rho, \rho_0)$ we know that $0 < \eta < \rho_0$, then from condition 3 we have $\|\boldsymbol{x}_1(\tau_k) + U(k, \boldsymbol{x}_1(\tau_k))\| < \rho$. Therefore there exists a $t_2 \in (\tau_k, t_1]$ such that $\eta \leq \|\boldsymbol{x}_1(t_2)\| < \rho$. Followed conditions 2 and 3 and Theorem 3.1.2 we have
   $$V(t, \boldsymbol{x}_1) \leq w_{\max}(t, t_0, w_0), \quad w_0 = \alpha(\|\boldsymbol{x}_0\|), \quad t \in [t_0, t_2] \tag{3.13}$$
   where $w_{\max}(t, t_0, w_0)$ is the maximal solution of (3.3). Followed condition 4 and recall that any solution of (3.3), $w(t, t_0, w_0) < \beta(\eta)$, we have
   $$\beta(\eta) \leq \beta(\|\boldsymbol{x}_1(t_2)\|) \leq V(t_2, \boldsymbol{x}_1(t_2)) \leq w_{\max}(t_2, t_0, w_0) < \beta(\eta) \tag{3.14}$$
   which leads to a contradiction. Hence, Claim 3.1.3 is true and the trivial solution of system (3.1) is *stable.*
2. Uniformly stable: Let us suppose that the trivial solution of (3.3) is uniformly stable, then ε is independent of t_0 and therefore following the same procedure of the stable case we can prove that the trivial solution of the system (3.1) is *uniformly stable.*
3. Asymptotically stable: Let us suppose that the trivial solution of (3.3) is asymptotically stable from which we know that the trivial solution of (3.3) is attractive and stable. Following the proof of stable case, we know that the trivial solution of (3.1) is stable. Let $\varepsilon_3 = \varepsilon(t_0, \min(\rho, \rho_0))$ then the stable property leads to
   $$\|\boldsymbol{x}_0\| < \varepsilon_3 \text{ implies } \|\boldsymbol{x}(t)\| < \rho, \quad t \geq t_0. \tag{3.15}$$
   We then need to prove the attractive property. Let $0 < \eta < \min(\rho, \rho_0)$, $t_0 \in \mathbb{R}_+$ and because the trivial solution of (3.3) is attractive, there exist $\varepsilon_4 = \varepsilon_4(t_0) > 0$ and $T = T(t_0, \eta) > 0$ such that

$$0 \le w_0 < \varepsilon_4 \text{ implies } w(t, t_0, w_0) < \beta(\eta), \quad t \ge t_0 + T. \tag{3.16}$$

Let us choose $\varepsilon_0 = \min(\varepsilon_3, \varepsilon_4)$ and let $\|\boldsymbol{x}_0\| < \varepsilon_0$, then from (3.15) we know that

$$\|\boldsymbol{x}(t)\| < \rho, \quad t \ge t_0,$$

from this fact and use the same process that leading to (3.13) we have

$$V(t, \boldsymbol{x}(t)) \le w_{\max}(t, t_0, \alpha(\|\boldsymbol{x}_0\|)), \qquad t \ge t_0 \tag{3.17}$$

from which we have

$$\begin{aligned}\beta(\|\boldsymbol{x}(t)\|) &\le V(t, \boldsymbol{x}(t)) \\ &\le w_{\max}(t, t_0, \alpha(\|\boldsymbol{x}_0\|)) < \beta(\eta), \quad t \ge t_0 + T\end{aligned} \tag{3.18}$$

which proves the trivial solution of (3.1) is attractive. Therefore the trivial solution of (3.1) is *asymptotically stable*.

4. Uniformly asymptotically stable: Let us suppose that the trivial solution of (3.3) is uniformly asymptotically stable, then ε_0 and T are independent of t_0 and therefore following the same procedure of asymptotically stable case we can prove that the trivial solution of the system (3.1) is *uniformly asymptotically stable*.

■

Remark 3.1.1. The system in (3.3) is a scalar impulsive differential equation. If we can find the comparison system of a high order impulsive control system, then the stability analysis may be simplified to that of a scalar impulsive differential equation. ♦

Theorem 3.1.4. *Let* $g(t, w) = \dot{\lambda}(t)w$, $\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$, $\dot{\lambda}(t) \ge 0$, $\psi_k(w) = d_k w$, $d_k \ge 0$ *for all* k, *then the origin of system (3.1) is*

1. *stable if*

$$\lambda(\tau_{k+1}) + \ln(d_k) \le \lambda(\tau_k), \text{ for all } k \tag{3.19}$$

is satisfied.

2. *asymptotically stable if*

$$\lambda(\tau_{k+1}) + \ln(\gamma d_k) \le \lambda(\tau_k), \text{ for all } k, \text{ where } \gamma > 1 \tag{3.20}$$

is satisfied.

⊠

Proof.

1. Stable: Let $w(t, t_0, w_0)$ be any solution of the following comparison system:

$$\begin{aligned} \dot{w} &= \dot{\lambda}(t)w, \quad t \neq \tau_k, \\ w(\tau_k^+) &= d_k w(\tau_k), \\ w(t_0^+) &= w_0 \geq 0 \end{aligned} \tag{3.21}$$

then we have

$$w(t, t_0, w_0) = w_0 \prod_{t_0 < \tau_k < t} d_k e^{\lambda(t) - \lambda(t_0)}, \quad t \geq t_0. \tag{3.22}$$

From $\dot{\lambda}(t) \geq 0$ we know that $\lambda(t)$ is nondecreasing and from (3.19), in the case of $0 < t_0 < \tau_1$, we have

$$w(t, t_0, w_0) \leq w_0 e^{\lambda(\tau_1) - \lambda(t_0)}, \quad t \geq t_0. \tag{3.23}$$

Then for a given $\eta > 0$ we can choose $\varepsilon = \eta / 2e^{\lambda(\tau_1) - \lambda(t_0)}$ such that $0 \leq w_0 < \varepsilon$ implies $w(t, t_0, w_0) \leq \eta/2 < \eta$. Hence we proved that the trivial solution of system (3.21) is stable. Followed Theorem 3.1.3 we know that the origin of system (3.1) is stable.

2. Asymptotically stable: If (3.20) holds then by using similar process we have

$$w(t, t_0, w_0) \leq \frac{w_0}{\gamma^k} e^{\lambda(\tau_1) - \lambda(t_0)}, \quad t \in (\tau_{k-1}, \tau_k] \tag{3.24}$$

from which we know $\lim_{t \to \infty} w(t, t_0, w_0) = 0$. Hence we proved that the trivial solution of system (3.21) is asymptotically stable. Followed Theorem 3.1.3 we know that the origin of system (3.1) is asymptotically stable.

■

Let the plant be the following nonlinear system:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{x}) \\ \boldsymbol{y} = C\boldsymbol{x} \end{cases} \tag{3.25}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, A is an $n \times n$ constant matrix, $\boldsymbol{y} \in \mathbb{R}^m$ is the output and C is an $m \times n$ constant matrix. $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The control instants are given by τ_k, $k = 1, 2, \cdots$. Then the nonlinear impulsive control system is given by

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{x}), \\ \boldsymbol{y} = C\boldsymbol{x}, & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{y}, & t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \tag{3.26}$$

where B is an $n \times m$ constant matrix. Then the impulsive control system can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{x}), \ t \neq \tau_k, \\ \Delta\boldsymbol{x} = BC\boldsymbol{x}, \qquad t = \tau_k, \quad k = 1, 2, \cdots, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0. \end{cases} \tag{3.27}$$

From which we know that $U(k, \boldsymbol{x}) = BC\boldsymbol{x}$.

Theorem 3.1.5. *Let the $n \times n$ matrix Γ be symmetric and positive definite, and $\lambda_1 > 0$, $\lambda_2 > 0$ are respectively, the smallest and the largest eigenvalues of Γ. Let*

$$Q = \Gamma A + A^\top \Gamma \tag{3.28}$$

and λ_3 be the largest eigenvalue of $\Gamma^{-1}Q$. $\boldsymbol{\phi}(\boldsymbol{x})$ is continuous and $\|\boldsymbol{\phi}(\boldsymbol{x})\| \leq L\|\boldsymbol{x}\|$ where $L > 0$ is a constant. λ_4 is the largest eigenvalue of the matrix

$$\Gamma^{-1}[I + (BC)^\top]\Gamma(I + BC). \tag{3.29}$$

Then the origin of impulsive control system (3.27) is asymptotically stable if

$$\left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right)(\tau_{k+1} - \tau_k) \leq -\ln(\gamma\lambda_4), \qquad \gamma > 1. \tag{3.30}$$

⊠

Proof. Let us construct a Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{x}^\top \Gamma \boldsymbol{x}$, when $t \neq \tau_k$ we have

$$\begin{aligned} D^+V(\boldsymbol{x}) &= \boldsymbol{x}^\top(A^\top\Gamma + \Gamma A)\boldsymbol{x} + [\boldsymbol{\phi}^\top(\boldsymbol{x})\Gamma\boldsymbol{x} + \boldsymbol{x}^\top\Gamma\boldsymbol{\phi}(\boldsymbol{x})] \\ &= \boldsymbol{x}^\top Q\boldsymbol{x} + [\boldsymbol{\phi}^\top(\boldsymbol{x})\Gamma\boldsymbol{x} + \boldsymbol{x}^\top\Gamma\boldsymbol{\phi}(\boldsymbol{x})] \\ &\leq \left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right)V(\boldsymbol{x}), \qquad t \neq \tau_k. \end{aligned} \tag{3.31}$$

Hence, condition 2 of Theorem 3.1.3 is satisfied with $g(t, w) = \left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right)w$.

Since

$$\begin{aligned} \|\boldsymbol{x} + U(k, \boldsymbol{x})\| &= \|\boldsymbol{x} + BC\boldsymbol{x}\| \\ &\leq \|I + BC\|\|\boldsymbol{x}\| \end{aligned} \tag{3.32}$$

and $\|I + BC\|$ is finite we know that there exists a $\rho_0 > 0$ such that $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$ implies that $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_\rho$ for all k.

When $t = \tau_k$, we have

$$\begin{aligned} V(\boldsymbol{x}+BC\boldsymbol{x})|_{t=\tau_k} &= (\boldsymbol{x}+BC\boldsymbol{x})^\top \Gamma(\boldsymbol{x}+BC\boldsymbol{x}) \\ &= \boldsymbol{x}^\top[I+(BC)^\top]\Gamma(I+BC)\boldsymbol{x} \\ &\le \lambda_4 V(\boldsymbol{x}). \end{aligned} \tag{3.33}$$

Hence condition 3 of Theorem 3.1.3 is satisfied with $\psi_k(w) = \lambda_4 w$. And we have

$$\lambda_1\|\boldsymbol{x}\|^2 \le V(\boldsymbol{x}) \le \lambda_2\|\boldsymbol{x}\|^2. \tag{3.34}$$

From which one can see that condition 4 of Theorem 3.1.3 is also satisfied with $\beta(x) = \lambda_1 x$ and $\alpha(x) = \lambda_2 x$. It follows from Theorem 3.1.3 that the asymptotic stability of the impulsive control system (3.27) is implied by that of the following comparison system:

$$\begin{cases} \dot{w}(t) = \left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right) w(t),\ t \ne \tau_k, \\ w(\tau_k^+) = \lambda_4 w(\tau_k), \\ w(t_0^+) = w_0 \ge 0. \end{cases} \tag{3.35}$$

It follows from Theorem 3.1.4 that if

$$\int_{\tau_k}^{\tau_{k+1}} \left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right) dt + \ln(\gamma\lambda_4) \le 0, \qquad \gamma > 1 \tag{3.36}$$

i.e.,

$$\left(\lambda_3 + 2L\sqrt{\frac{\lambda_2}{\lambda_1}}\right)(\tau_{k+1}-\tau_k) \le -\ln(\gamma\lambda_4), \qquad \gamma > 1 \tag{3.37}$$

is satisfied, then the origin of (3.27) is asymptotically stable. ■

We then generalize Theorem 3.1.5 for the following time-varying impulsive system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + \boldsymbol{\phi}(t,\boldsymbol{x}), \quad t \ne \tau_k, \\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x}, \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0 \end{aligned} \qquad (3.38)}$$

where $A(t)$ is a continuous $n \times n$ matrix, $B_k, k = 1, 2, \cdots$, are $n \times n$ matrices. In this case, the impulsive control law is linear; namely, $U(k, \boldsymbol{x}) = B_k\boldsymbol{x}$.

Theorem 3.1.6. *Suppose that*

1. *There is a self adjoint and positive matrix $\Gamma(t)$ that is continuously differentiable. Let $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ be the smallest and the largest eigenvalues of $\Gamma(t)$;*

2. Let

$$Q(t) = \frac{d\Gamma(t)}{dt} + \Gamma(t)A(t) + A^\top(t)\Gamma(t) \tag{3.39}$$

and $\mu_{\max} \in C[\mathbb{R}_+, \mathbb{R}_+]$ *be the largest eigenvalue of the matrix* $\Gamma^{-1}(t)Q(t)$;

3. $\boldsymbol{\phi} \in C[\mathbb{R}_+ \times \mathcal{S}_\rho, \mathbb{R}^n]$ *and* $\|\boldsymbol{\phi}(t, \boldsymbol{x})\| \leq L(t)\|\boldsymbol{x}\|$ *on* $\mathbb{R}_+ \times \mathcal{S}_\rho$ *where* $\beta \in C[\mathbb{R}_+, \mathbb{R}_+]$ *and let*

$$\sigma(t) = \mu_{\max}(t) + 2L(t)\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}};$$

4. $\psi_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ *is the largest eigenvalue of the matrix*

$$\Gamma^{-1}(t)(I + B_k^\top)\Gamma(t)(I + B_k), \text{ for all } k; \tag{3.40}$$

Then the origin of the system (3.38) is asymptotically stable if

$$\int_{\tau_k}^{\tau_{k+1}} \sigma(s)ds + \ln[\gamma\psi_k(\tau_k)] \leq 0, \quad \gamma > 1. \tag{3.41}$$

⊠

Proof. Let us construct a Lyapunov function

$$V(t, \boldsymbol{x}) = \boldsymbol{x}^\top \Gamma(t)\boldsymbol{x},$$

when $t \neq \tau_k$ we have

$$\begin{aligned} D^+V(t, \boldsymbol{x}) &= \boldsymbol{x}^\top [A^\top(t)\Gamma(t) + \Gamma(t)A(t)]\boldsymbol{x} + [\boldsymbol{\phi}^\top(t, \boldsymbol{x})\Gamma(t)\boldsymbol{x} + \boldsymbol{x}^\top \Gamma(t)\boldsymbol{\phi}(t, \boldsymbol{x})] \\ &= \boldsymbol{x}^\top Q(t)\boldsymbol{x} + [\boldsymbol{\phi}^\top(t, \boldsymbol{x})\Gamma(t)\boldsymbol{x} + \boldsymbol{x}^\top \Gamma(t)\boldsymbol{\phi}(t, \boldsymbol{x})] \\ &\leq \sigma(t)V(t, \boldsymbol{x}), \qquad t \neq \tau_k. \end{aligned} \tag{3.42}$$

Hence, condition 2 of Theorem 3.1.3 is satisfied with $g(t, w) = \sigma(t)w$.

Since

$$\|\boldsymbol{x} + B_k\boldsymbol{x}\| \leq \|I + B_k\|\|\boldsymbol{x}\| \tag{3.43}$$

and $\|I + B_k\|$ is finite for all k, then there is a $\rho_0 > 0$ such that $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$ implies $\boldsymbol{x} + B_k\boldsymbol{x} \in \mathcal{S}_\rho$.

When $t = \tau_k$, we have

$$\begin{aligned} V(t, \boldsymbol{x} + B_k\boldsymbol{x})|_{t=\tau_k} &= (\boldsymbol{x} + B_k\boldsymbol{x})^\top \Gamma(t)(\boldsymbol{x} + B_k\boldsymbol{x}) \\ &= \boldsymbol{x}^\top (I + B_k^\top)\Gamma(t)(I + B_k)\boldsymbol{x} \\ &\leq \psi_k(\tau_k)V(t, \boldsymbol{x}). \end{aligned} \tag{3.44}$$

Hence condition 3 of Theorem 3.1.3 is satisfied with $\psi_k(w) = \psi_k w$. And we have

$$\lambda_{\min}\|\boldsymbol{x}\|^2 \le V(\boldsymbol{x}) \le \lambda_{\max}\|\boldsymbol{x}\|^2. \tag{3.45}$$

From which one can see that condition 4 of Theorem 3.1.3 is also satisfied with $\beta(x) = \lambda_{\min}x$ and $\alpha(x) = \lambda_{\max}x$. It follows from Theorem 3.1.3 that the asymptotic stability of the impulsive control system (3.38) is implied by that of the following comparison system:

$$\begin{aligned} \dot{w} &= \sigma(t)w, \quad t \neq \tau_k, \\ w(\tau_k^+) &= \psi_k(\tau_k)w(\tau_k), \\ w(t_0^+) &= w_0 \ge 0. \end{aligned} \tag{3.46}$$

It follows from (3.41) and the second conclusion of Theorem 3.1.4 we know that the trivial solution of (3.46) is asymptotically stable. Therefore, the trivial solution of (3.38) is asymptotically stable. ■

Remark 3.1.2. We have the following special cases:

1. When $\sigma(t) = \sigma > 0$, then condition (3.41) becomes
$$\psi_k(\tau_k) \le \frac{\sigma}{\gamma}e^{\tau_k - \tau_{k+1}};$$
2. Let $\sigma(t) = \frac{1}{t+c} > 0$, then condition (3.41) becomes
$$\psi_k(\tau_k) \le \frac{\tau_k + c}{\gamma(\tau_{k+1} + c)};$$
3. Let $\sigma(t) = c + \sin(t) > 0$, then condition (3.41) becomes
$$\psi_k(\tau_k) \le \frac{1}{\gamma}\exp[c(\tau_k - \tau_{k+1}) + \cos(\tau_{k+1}) - \cos(\tau_k)].$$

♦

Since the usefulness of the stability of comparison systems, we present some results of this kind as follows. Let us consider the following comparison system:

$$\begin{aligned} \dot{w} &= -\sigma(t)\chi(w), \quad t \neq \tau_k, \\ w(\tau_k^+) &= \psi_k(w(\tau_k)), \\ w(t_0^+) &= w_0 \ge 0 \end{aligned} \tag{3.47}$$

where $t, w \in \mathbb{R}_+$, $k \in \mathbb{N}$, $\sigma \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}_+]$, $\psi_k, \chi \in \mathcal{K}$.

Theorem 3.1.7. *Assume that there is such a $\varrho > 0$ that for $\theta \in (0, \varrho]$ and $k \in \mathbb{N}$*

$$-\int_{\tau_k}^{\tau_{k+1}} \sigma(s)ds + \int_{\theta}^{\psi_{k+1}(\theta)} \frac{1}{\chi(s)}ds \le -\varpi_{k+1}, \quad \varpi_{k+1} \ge 0. \tag{3.48}$$

Then the trivial solution of the comparison system (3.47) is

1. *stable;*
2. *asymptotically stable if*

$$\sum_{k=1}^{\infty} \varpi_k = \infty. \tag{3.49}$$

⊠

Proof. Let us assume that $0 \le t_0 < \tau_1$ and choose $\delta = \delta(t_0) > 0$ such that if $w_0 \in (0, \delta)$ then $\psi_1(w_0) < \varrho$. Let $w(t) = w(t, t_0, w_0)$ be a solution of system (3.47), then $w(t)$ is nonincreasing in $(\tau_k, \tau_{k+1}], k \in \mathbb{N}$ and since $\psi_1 \in \mathcal{K}$ we have

$$w(\tau_1^+) = \underbrace{\psi_1(w(\tau_1)) \le \psi_1(w_0)}_{w(\tau_1) \le w_0} < \varrho. \tag{3.50}$$

Since $w(t)$ is nonincreasing in $t \in (\tau_1, \tau_2]$, it follows from (3.50) that

$$w(\tau_2) \le w(\tau_1^+) < \varrho. \tag{3.51}$$

It follows from (3.47) that for $t \in (\tau_k, \tau_{k+1}], k \in \mathbb{N}$ we have

$$\int_{w(\tau_k^+)}^{w(t)} \frac{1}{\chi(s)} ds = -\int_{\tau_k}^{t} \sigma(s) ds. \tag{3.52}$$

From (3.52) it follows that

$$\int_{w(\tau_1^+)}^{w(\tau_2)} \frac{1}{\chi(s)} ds = -\int_{\tau_1}^{\tau_2} \sigma(s) ds. \tag{3.53}$$

It follows from (3.48) and (3.51) that

$$\int_{w(\tau_2)}^{w(\tau_2^+)} \frac{1}{\chi(s)} ds = \int_{w(\tau_2)}^{\psi_2(w(\tau_2))} \frac{1}{\chi(s)} ds \le \int_{\tau_1}^{\tau_2} \sigma(s) ds - \varpi_2. \tag{3.54}$$

It follows from (3.53) and (3.54) that

$$\int_{w(\tau_1^+)}^{w(\tau_2^+)} \frac{1}{\chi(s)} ds = \int_{w(\tau_1^+)}^{w(\tau_2)} \frac{1}{\chi(s)} ds + \int_{w(\tau_2)}^{w(\tau_2^+)} \frac{1}{\chi(s)} ds \le -\varpi_2 \tag{3.55}$$

from which and since $\chi \in \mathcal{K}$ we know that $w(\tau_2^+) \le w(\tau_1^+) < \varrho$. Using mathematical induction we can prove that

$$\int_{w(\tau_k^+)}^{w(\tau_{k+1}^+)} \frac{1}{\chi(s)} ds \le -\varpi_{k+1} \tag{3.56}$$

and $w(\tau_{k+1}^+) \le w(\tau_k^+)$ which yields $w(\tau_k^+) < \varrho$ for all $k \in \mathbb{N}$. Since $w(t)$ is nonincreasing in $(\tau_k, \tau_{k+1}]$, $k \in \mathbb{N}$, we immediately know that if $w_0 < \delta$ then $w(t) < \varrho$ for all $t > t_0$. This proves the stability of the trivial solution.

To prove the asymptotic stability we only need to prove the following claim:
Claim 3.1.7:

$$\lim_{k\to\infty} w_k = 0.$$

If Claim 3.1.7 is false, then there is such a $\eta > 0$ that $w(\tau_k^+) \ge \eta$ for $k \in \mathbb{N}$. Since the sequence $\{w(\tau_k^+)\}$ is nonincreasing and $\chi \in \mathcal{K}$ we have

$$\chi(\eta) \le \chi(w(\tau_k^+)) \le \chi(w(\tau_{k+1}^+)). \tag{3.57}$$

And it follows from (3.56) and (3.57) that

$$\underbrace{\varpi_k \le \int_{w(\tau_k^+)}^{w(\tau_{k-1}^+)}}_{(3.56)} \underbrace{\frac{1}{\chi(s)}ds \le \frac{w(\tau_{k-1}^+) - w(\tau_k^+)}{\chi(\eta)}}_{(3.57)}. \tag{3.58}$$

It follows from (3.58) that

$$\begin{aligned} w(\tau_k^+) &\le w(\tau_{k-1}^+) - \chi(\eta)\varpi_k, \\ w(\tau_{k+i}^+) &\le w(\tau_k^+) - \chi(\eta)\sum_{j=k+1}^{k+i} \varpi_j. \end{aligned} \tag{3.59}$$

It follows from (3.59) and (3.49) that we have the following contradiction to the fact that $w(t) \ge 0$:

$$\lim_{i\to\infty} w(\tau_{k+i}^+) = -\infty.$$

Therefore Claim 3.1.7 is true and this finishes the proof. ■

Theorem 3.1.8. *Assume that*

1. *there is such a* $\varrho > 0$ *that for* $\theta \in (0, \varrho]$ *and* $k \in \mathbb{N}$

$$-\int_{\tau_k}^{\tau_{k+1}} \sigma(s)ds + \int_{\theta}^{\psi_{k+1}(\theta)} \frac{1}{\chi(s)}ds \le -\varpi_{k+1}, \quad \varpi_{k+1} \ge 0; \tag{3.60}$$

2. *there is a* $\psi \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi(s) > 0$ *if* $s > 0$, $\psi(0) = 0$, *such that* $\psi_k(s) \le \psi(s)$ *for* $s \in (0, \varrho]$ *and* $k \in \mathbb{N}$.

Then the trivial solution of the comparison system (3.47) is

1. *uniformly stable;*

2. *uniformly asymptotically stable if for any* $K > 0$ *there is such a* $\phi > 0$ *that for* $t_0 \in \mathbb{R}_+$ *and* $t \geq t_0 + \phi$

$$\sum_{t_0<\tau_k<t}^{\infty} \varpi_k > K.$$

⊠

Proof. We first prove the uniform stability. From the proof of the first conclusion of Theorem 3.1.7 we know that the trivial solution is stable. Then it follows from condition 2 that we can choose such a $\delta > 0$ independent of t_0 that $\psi(w_0) < \varrho$ if $w_0 \in (0, \delta)$. Therefore, the trivial solution is uniformly stable because for any $t_0 \in (\tau_k, \tau_{k+1}]$, $k \in \mathbb{N}$, we have $\psi_{k+1}(w_0) \leq \psi(w_0) < \varrho$ (this leads to $w(t) < \varrho$ for all $t > t_0$).

We are now going to prove the uniform asymptotic stability. Let us choose such a $\delta > 0$ that $\psi(s) < \varrho$ for $s \in [0, \delta)$. Let $w(t) = w(t, t_0, w_0)$ with $t_0 \in \mathbb{R}_+$ and $w(t_0, t_0, w_0) = w_0 \in [0, \delta)$, then for $t > t_0$ we have $w(t) < \varrho$ because the trivial solution is uniformly stable. Let us choose an arbitrary number $\xi \in (0, \delta)$, then it follows from the last assumption that there is a $\phi > 0$ such that for any $t_0 \in \mathbb{R}_+$ and $t \geq t_0 + \phi$ we have

$$\sum_{t_0<\tau_k<t} \varpi_k - \varpi_{\nu(t_0,t)} > \frac{\varrho}{\chi(\xi)} \tag{3.61}$$

where $\nu(t_0, t) \triangleq \min\{i \in \mathbb{N} |\ \tau_i \in (t_0, t)\}$. It follows from (3.56) that for an $m \in \mathbb{N}$

$$\begin{aligned}\int_{w(\tau_{j+m}^+)}^{w(\tau_j^+)} \frac{1}{\chi(s)} ds &= \int_{w(\tau_{j+1}^+)}^{w(\tau_j^+)} \frac{1}{\chi(s)} ds + \int_{w(\tau_{j+2}^+)}^{w(\tau_{j+1}^+)} \frac{1}{\chi(s)} ds + \cdots \\ &\quad + \int_{w(\tau_{j+m}^+)}^{w(\tau_{j+m-1}^+)} \frac{1}{\chi(s)} ds = \sum_{i=1}^{m} \varpi_{j+i}.\end{aligned} \tag{3.62}$$

Assume that $t_0 \in [\tau_{j-1}, \tau_j)$ and $t_0 + \phi \in (\tau_k, \tau_{k+1})$ and $w_0 \in [0, \delta)$, then we have the following claim:

Claim 3.1.8: $w(\tau_i^+)$, $i = j, j+1, \cdots, k$ are not greater than ξ.

If Claim 3.1.8 is not true, then we have

$$\begin{aligned}\frac{\varrho - w(\tau_k^+)}{\chi(\xi)} &= \underbrace{\int_{w(\tau_k^+)}^{\varrho} \frac{1}{\chi(\xi)} ds > \int_{w(\tau_k^+)}^{w(\tau_j^+)}}_{w(t)<\varrho} \underbrace{\frac{1}{\chi(\xi)} ds > \int_{w(\tau_k^+)}^{w(\tau_j^+)} \frac{1}{\chi(s)} ds}_{w(\tau_i^+)>\xi, i=j,j+1,\cdots,k.} \\ &\geq \sum_{t_0<\tau_i<t_0+\phi} \varpi_i - \varpi_{\nu(t_0,t_0+\phi)} \qquad \Leftarrow (3.62) \\ &> \frac{\varrho}{\chi(\xi)} \qquad \Leftarrow (3.61)\end{aligned} \tag{3.63}$$

which leads to $w(\tau_k^+) < 0$. This is a contradiction. Therefore, Claim 3.1.8 is true. This finishes the proof. ∎

3.2 Impulsive Control of Chaotic Systems

We then present some design examples to show how to perform impulsive control on chaotic dynamical systems. Impulsive control of chaotic systems has important applications to secure communication systems and spread spectrum communication systems. Since Lorenz system is a benchmark of chaotic systems, in this section we present the impulsive control of Lorenz system.

3.2.1 Theory

The Lorenz (chaotic) system[21] is given by

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \tag{3.64}$$

where σ, r, and b are three real positive parameters. When these parameters are chosen as $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$, the Lorenz system is chaotic.

By decomposing the linear and nonlinear parts of the Lorenz system in (3.64), we can rewrite it as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \Phi(\boldsymbol{x}) \tag{3.65}$$

where $\boldsymbol{x}^\top = (x, y, z)$ and

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad \Phi(\boldsymbol{x}) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}. \tag{3.66}$$

The impulsively controlled Lorenz system is then given by

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \Phi(\boldsymbol{x}), & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x}, & t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \tag{3.67}$$

where τ_k denote the moments when impulsive control occurs. Without loss of generality, we assume that $\tau_1 > t_0$. We say $\{\tau_i\}$ is *equidistant* if exists a constant $\delta > 0$ such that $\tau_{i+1} - \tau_i = \delta$, $i = 1, 2, \cdots$. Then we use the following theorem to guarantee that the impulsively controlled Lorenz system is asymptotically stabilized at origin.

Theorem 3.2.1. *Let d_1 be the largest eigenvalue of $(I + B^\top)(I + B)$. Let q be the largest eigenvalue of $(A + A^\top)$ and impulses be equidistant with an interval $\delta > 0$. If*

$$0 \leq q \leq -\frac{1}{\delta}\ln(\xi d_1), \quad \xi > 1 \tag{3.68}$$

then the origin of the impulsively controlled Lorenz system (3.67) is asymptotically stable. ⊠

Proof. Let $V(t, \boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{x}$. For $t \neq \tau_i$, we have

$$
\begin{aligned}
D^+ V(t, \boldsymbol{x}) &= \boldsymbol{x}^\top A \boldsymbol{x} + \boldsymbol{x}^\top A^\top \boldsymbol{x} + \boldsymbol{x}^\top \Phi(\boldsymbol{x}) + \Phi^\top(\boldsymbol{x}) \boldsymbol{x} \\
&\leq q \boldsymbol{x}^\top \boldsymbol{x} + 2(-xyz + xyz) \\
&= qV(t, \boldsymbol{x}).
\end{aligned}
\tag{3.69}
$$

Hence condition 2 of Theorem 3.1.3 is satisfied with $g(t, w) = qw$. Since

$$
\begin{aligned}
\|\boldsymbol{x} + U(i, \boldsymbol{x})\| &= \|\boldsymbol{x} + B\boldsymbol{x}\| \\
&\leq \|I + B\| \|\boldsymbol{x}\|
\end{aligned}
\tag{3.70}
$$

and $\|I + B\|$ is finite, this is a $\rho_0 > 0$ and $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$ such that $\boldsymbol{x} + U(i, \boldsymbol{x}) \in \mathcal{S}_\rho$.

For $t = \tau_i$ we have

$$
\begin{aligned}
V(\tau_i, \boldsymbol{x} + B\boldsymbol{x}) &= (\boldsymbol{x} + B\boldsymbol{x})^\top (\boldsymbol{x} + B\boldsymbol{x}) \\
&= \boldsymbol{x}^\top (I + B^\top)(I + B) \boldsymbol{x} \\
&\leq d_1 V(\tau_i, \boldsymbol{x}).
\end{aligned}
\tag{3.71}
$$

Hence condition 3 of Theorem 3.1.3 is satisfied with $\psi_i(w) = d_1 w$. We can see that condition 4 of Theorem 3.1.3 is also satisfied. It follows from Theorem 3.1.3 that the asymptotic stability of system (3.67) is implied by that of the following comparison system:

$$
\begin{cases}
\dot{\omega} = q\omega, \ \ t \neq \tau_i, \\
\omega(\tau_i) = d_1 \omega(\tau_i), \\
\omega(t_0^+) = \omega_0 \geq 0.
\end{cases}
\tag{3.72}
$$

Since $0 \leq q \leq -\frac{1}{\delta} \ln(\xi d_1)$, we have

$$
\int_{\tau_i}^{\tau_{i+1}} q dt + \ln(\xi d_1) \leq 0, \qquad \xi > 1, \qquad \text{for all } i \tag{3.73}
$$

It follows from Theorem 3.1.4 that the trivial solution of (3.67) is asymptotically stable. ∎

The above Theorem also gives an estimation of the upper boundary of δ, $\delta_{\max}$:

$$
\delta_{\max} = -\frac{\ln(\xi d_1)}{q} > 0, \quad \xi \to 1^+. \tag{3.74}
$$

From above we can see that d_1 should satisfy $0 < d_1 < 1$.

3.2.2 Simulation Results

We then use some simulation results to show how the impulsive control scheme works.

Example 3.2.1. In this example, we choose the matrix B as

$$B = \begin{pmatrix} k & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.75}$$

then the impulsively controlled Lorenz system is given by

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \ , \qquad t \neq \tau_i, \\ \dot{z} = xy - bz \end{cases}$$
$$\Delta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad t = \tau_i. \tag{3.76}$$

We have

$$(I + B^\top)(I + B) = \begin{pmatrix} (k+1)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.77}$$

whose largest eigenvalue is given by

$$d_1 = \max(1, (k+1)^2). \tag{3.78}$$

Since $d_1 \in (0, 1)$ should be satisfied, from above we know that $k \in (-2, 0)$.

Let the parameters be $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$, then we have

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad A + A^\top = \begin{pmatrix} -20 & 38 & 0 \\ 38 & -2 & 0 \\ 0 & 0 & -\frac{16}{3} \end{pmatrix}. \tag{3.79}$$

We find $q = 28.051249$. Then estimations of the boundaries of stable regions are given by

$$0 \leq \delta \leq -\frac{\ln \xi + \ln(k+1)^2}{q}, \quad k \in (-2, 0). \tag{3.80}$$

Figure 3.1 shows the stable region for different ξ's. The entire region under the curve of $\xi = 1$ is the stable region. When $\xi \to \infty$, the stable region approaches a vertical line $k = -1$.

The simulation results are shown in Fig. 3.2. The solid, dash-dotted, and dotted waveforms show $x(t)$, $y(t)$ and $z(t)$, respectively. Figure 3.2(a) shows the stable results within the stable region with $k = -1.5$ and $\delta = 0.04$. Observe that the system asymptotically approaches the origin with a settling time about 0.2. Figure 3.2(b) shows the unstable results with $k = -1.5$ and $\delta = 0.17$. ★

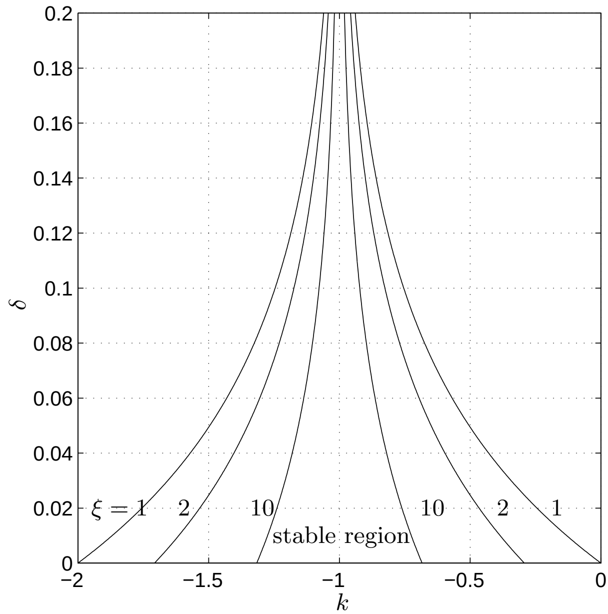


Fig. 3.1. The estimation of boundaries of stable region with different ξ's used in Example 3.2.1.

Example 3.2.2. In this example, we choose the matrix B as

$$B = \begin{pmatrix} k & 0 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & -0.6 \end{pmatrix}. \tag{3.81}$$

Similarly, k should satisfy $-2 < k < 0$. Then we have

$$d_1 = \begin{cases} (k+1)^2, & (k+1)^2 \geq 0.16, \\ 0.16, & \text{otherwise.} \end{cases} \tag{3.82}$$

Then an estimation of the boundaries of the stable region is given by

$$0 \leq \delta \leq \begin{cases} -\frac{\ln \xi + \ln (k+1)^2}{q}, & (k+1)^2 \geq 0.16 \\ -\frac{\ln \xi + \ln (0.16)}{q}, & \text{otherwise} \end{cases}, \quad -2 < k < 0. \tag{3.83}$$

Figure 3.3 shows the stable region for different ξ's. The entire region under the curve of $\xi = 1$ is the stable region.

The simulation results are shown in Fig. 3.4. The solid, dash-dotted, and dash-dotted waveforms show $x(t)$, $y(t)$ and $z(t)$, respectively. Figure 3.4(a) shows a stable result within the stable region with $k = -0.6$ and $\delta = 0.06$. Observe that the system asymptotically approaches the origin with a settling time around 1 time unit. Figure 3.4(b) shows an unstable result with $k = -0.6$ and $\delta = 0.09$. ★

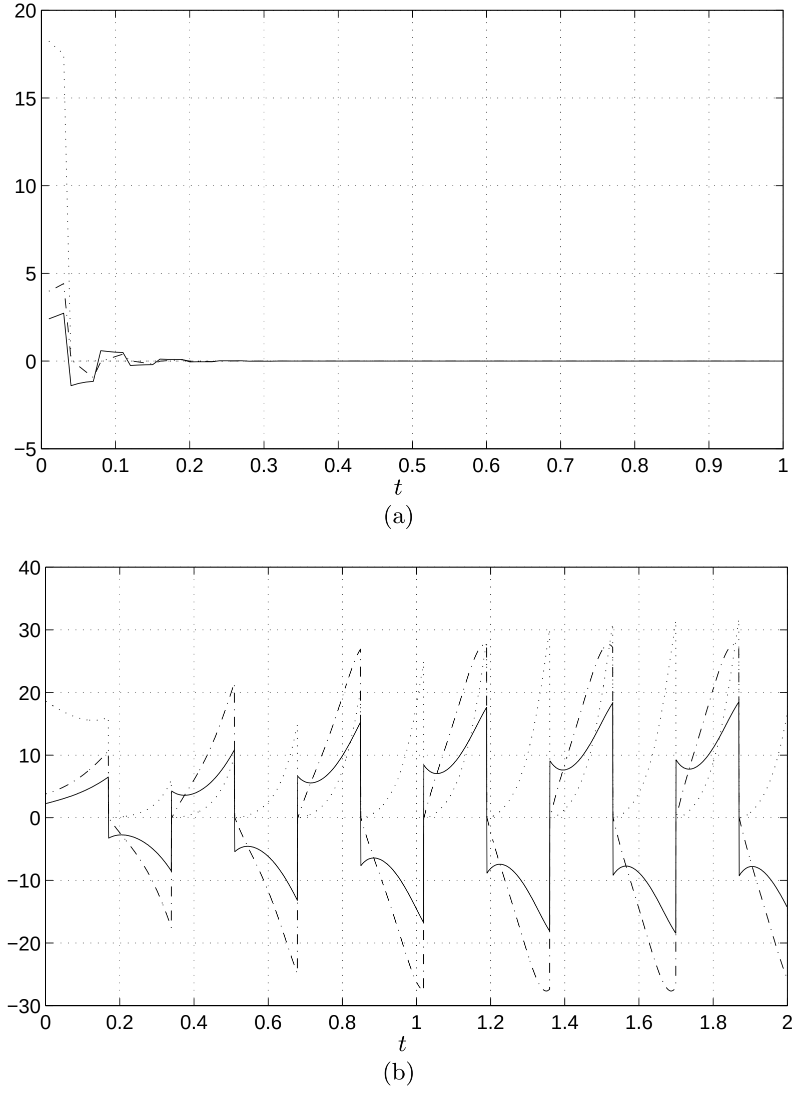


Fig. 3.2. Impulsive control of the Lorenz systems with strong control. (a) Stable results within stable region. (b) Unstable results outside stable region.

Note 3.2.1. The comparison method can be found in [37]. Impulsive control of chaotic systems can be found in [52, 48, 49]. ♦

3.3 Comparison Systems with Two Measures

Theorem 3.3.1. *Suppose that*

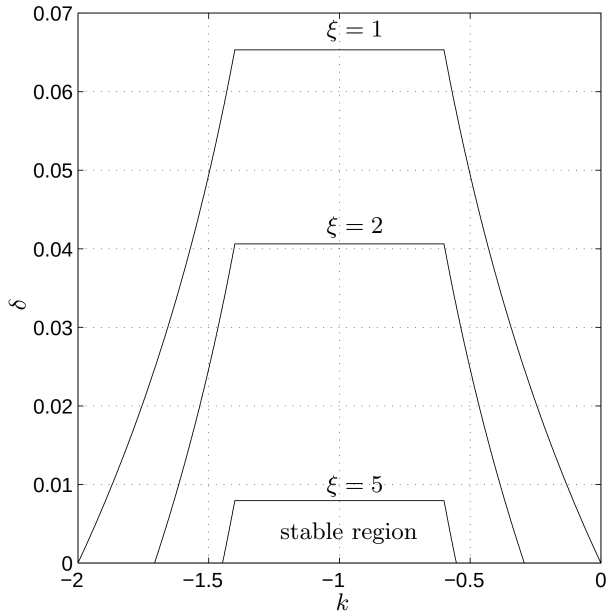


Fig. 3.3. The estimation of boundaries of stable region with different ξ's used in Experiment 3.2.2.

1. $\boldsymbol{f}(t, 0) = 0$, $\boldsymbol{u}(t, 0) = 0$, $g(t, 0) = 0$ *and* $U(k, 0) = 0$ *for all* k;
2. $h_0 \in \mathcal{H}$, $h \in \mathcal{H}$ *and* h_0 *is finer than* h;
3. $V \in \mathcal{V}_0$ *is* h*-positive definite and* h_0*-decrescent and*

$$D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x})), \quad t \neq \tau_k, \quad (t, \boldsymbol{x}) \in \mathcal{S}_\rho(h) \tag{3.84}$$

where $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is continuous in* $(\tau_{k-1}, \tau_k] \times \mathbb{R}$ *and for each* $x \in \mathbb{R}$, $k = 1, 2, \cdots$, *the limit*

$$\lim_{(t,y)\to(\tau_k^+,x)} g(t, y) = g(\tau_k^+, x)$$

exists.

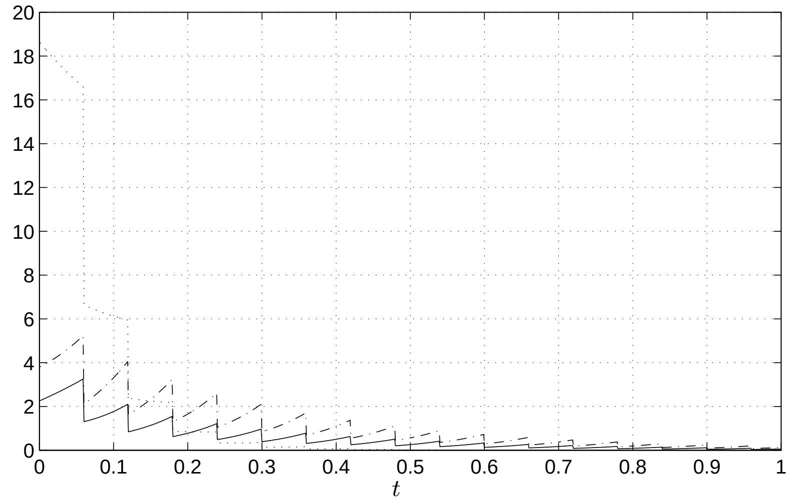
4.

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(\tau_k, \boldsymbol{x})), \quad k = 1, 2, \cdots, \tag{3.85}$$

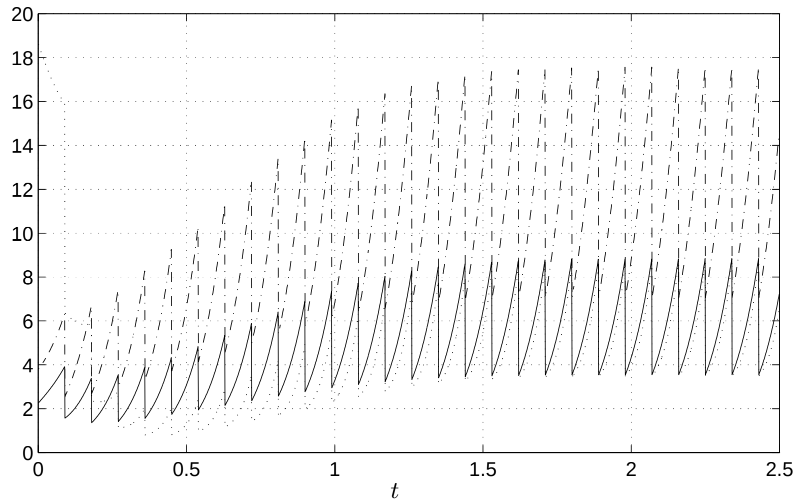
$\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ *is nondecreasing;*

5. *there is a* ρ_0, $0 < \rho_0 < \rho$, *such that* $h(\tau_k, \boldsymbol{x}) < \rho_0$ *implies* $h(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) < \rho$.

Then the stability properties of the trivial solution of the comparison system (3.3) imply the corresponding (h_0, h)*-stability properties of the trivial solution of (3.1).* ⊠



(a)



(b)

Fig. 3.4. Simulation results of impulsive control of the Lorenz system with weak control. (a) Stable results within stable region. (b) Unstable results outside stable region.

Proof.

1. (h_0, h)-stable: From the condition that $V(t, \boldsymbol{x})$ is h-positive definite it follows that there is a $\rho_1 \in (0, \rho]$ and $\beta \in \mathcal{K}$ such that

$$\beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}), \text{ whenever } h(t, \boldsymbol{x}) < \rho_1. \tag{3.86}$$

Let $0 < \eta < \min(\rho_0, \rho_1)$ and assume that the trivial solution of the comparison system (3.3) is stable, then given $\beta(\eta)$ there is a $\varepsilon_1 = \varepsilon_1(t_0, \eta) > 0$ such that for any solution, $w(t, t_0, w_0)$, of the comparison system (3.3)

$$w_0 < \varepsilon_1 \text{ implies } w(t, t_0, w_0) < \beta(\eta), \quad t \geq t_0. \tag{3.87}$$

Let us choose $w_0 = V(t_0, \boldsymbol{x}_0)$ and since $V(t, \boldsymbol{x})$ is h_0-decrescent and h_0 is finer than h, there are a $\rho_2 > 0$ and a function $\alpha \in \mathcal{K}$ such that if $h_0(t_0^+, \boldsymbol{x}_0) < \rho_2$ then we have

$$h(t_0^+, \boldsymbol{x}_0) < \rho_1 \text{ and } V(t_0^+, \boldsymbol{x}_0) \leq \alpha(h_0(t_0^+, \boldsymbol{x}_0)). \tag{3.88}$$

Then from (3.86) and (3.88) we know that if $h_0(t_0^+, \boldsymbol{x}_0) < \rho_2$ then

$$\beta(h(t_0^+, \underbrace{\boldsymbol{x}_0)) \leq V(t_0^+,}_{(3.86)} \underbrace{\boldsymbol{x}_0) \leq \alpha(h_0(t_0^+,}_{(3.88)} \boldsymbol{x}_0)). \tag{3.89}$$

Let us choose a $\varepsilon = \varepsilon(t_0, \eta) \in (0, \rho_2]$ such that $\alpha(\varepsilon) < \varepsilon_1$ and let $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$, then we have the following claim:
Claim 3.3.1: For any solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of system (3.1) we have

$$h(t, \boldsymbol{x}(t)) < \eta, \quad t \geq t_0, \text{ whenever } h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon. \tag{3.90}$$

If Claim 3.3.1 is not true, then there is a solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$ of (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$ and a $t_1 > t_0$ such that $t_1 \in (\tau_k, \tau_{k+1}]$ for some k satisfying

$$h(t_1, \boldsymbol{x}_1(t_1)) \geq \eta \text{ and } h(t, \boldsymbol{x}_1(t)) < \eta, \quad t \in [t_0, \tau_k]. \tag{3.91}$$

Since $0 < \eta < \min(\rho_0, \rho_1)$ we have $0 < \eta < \rho_0$ and from condition 5 we have

$$h(\tau_k^+, \boldsymbol{x}_1(\tau_k) + U(k, \boldsymbol{x}_1(\tau_k))) < \rho. \tag{3.92}$$

From (3.91) it follows that $h(\tau_k, \boldsymbol{x}_1(\tau_k)) < \eta$. Then, there is a $t_2 \in (\tau_k, t_1]$ such that

$$h(t_2, \boldsymbol{x}_1(t_2)) \in [\eta, \rho) \text{ and } h(t, \boldsymbol{x}_1(t)) < \rho \text{ for } t \in [t_0, t_2]. \tag{3.93}$$

From conditions 3 and 4, for $t \in [t_0, t_2]$ we have

$$\begin{aligned} D^+V(t, \boldsymbol{x}_1(t)) &\leq g(t, V(t, \boldsymbol{x}_1(t))), \quad t \neq \tau_j, \\ V(\tau_j^+, \boldsymbol{x}_1(\tau_j) + U(j, \boldsymbol{x}_1(\tau_j))) &\leq \psi_j(V(\tau_j, \boldsymbol{x}_1(\tau_j))), \\ & j = 1, 2, \cdots, k. \end{aligned} \tag{3.94}$$

Then from Theorem 3.1.2 we have

$$V(t, \boldsymbol{x}_1(t)) \leq w_{\max}(t, t_0, w_0), \quad t \in [t_0, t_2] \tag{3.95}$$

where $w_0 = V(t_0^+, \boldsymbol{x}_0)$. Then from (3.86) and (3.87) we have

$$\underbrace{\beta(\eta) \le \beta(h(t_2,}_{(3.93)}\underbrace{\boldsymbol{x}_1(t_2))) \le V(t_2,}_{(3.86)}\underbrace{\boldsymbol{x}_1(t_2)) \le w_{\max}(t_2,}_{(3.95)}\underbrace{t_0, w_0) < \beta(\eta)}_{(3.87)} \tag{3.96}$$

which leads to a contradiction. Therefore Claim 3.3.1 is true and the trivial solution of system (3.1) is (h_0, h)-stable.

2. (h_0, h)-asymptotically stable: Assume that the trivial solution of (3.3) is asymptotically stable, then from the first part of this proof we know that the trivial solution of system (3.1) is (h_0, h)-stable. Let us set $\varepsilon_0 = \varepsilon(t_0, \min(\rho_0, \rho_1))$, $0 < \eta < \min(\rho_0, \rho_1)$ and $t_0 \in \mathbb{R}_+$, then from the fact that the trivial solution of (3.3) is attractive we know there are a $\varepsilon_4 = \varepsilon(t_0) > 0$ and a $T = T(t_0, \eta) > 0$ such that

$$w_0 < \varepsilon_4 \text{ implies } w(t, t_0, w_0) < \beta(\eta), \quad t \ge t_0 + T. \tag{3.97}$$

Let us choose $w_0 = V(t_0^+, \boldsymbol{x}_0)$ and find a $\varepsilon_3 = \varepsilon_3(t_0) \in (0, \rho_2]$ such that $\alpha(\varepsilon_3) < \varepsilon_4$. Let $\varepsilon_5 = \min(\varepsilon_3, \varepsilon_0)$ and $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon_5$ then from $\varepsilon_5 \le \varepsilon_3 \le \rho_2$ we have $h_0(t_0^+, \boldsymbol{x}_0) < \rho_2$. Followed the same reasoning process leading to the first conclusion in (3.88) we know that for any solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of (3.1) we have $h(t, \boldsymbol{x}(t)) < \rho_1 \le \rho, t \ge t_0$; namely, $(t, \boldsymbol{x}(t)) \in \mathcal{S}_\rho(h)$. Then from conditions 3 and 4, for $t \ge t_0$ we have

$$\begin{aligned} D^+V(t, \boldsymbol{x}(t)) &\le g(t, V(t, \boldsymbol{x}(t))), \quad t \ne \tau_k, \\ V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) &\le \psi_k(V(\tau_k, \boldsymbol{x}(\tau_k))), \\ k &= 1, 2, \cdots . \end{aligned} \tag{3.98}$$

Then from Theorem 3.1.2 it follows that

$$V(t, \boldsymbol{x}(t)) \le w_{\max}(t, t_0, w_0), \quad t \ge t_0. \tag{3.99}$$

If the trivial solution of (3.1) is not (h_0, h_1)-attractive, then there is a sequence $\{t_i\}$ satisfying $t_i \ge t_0 + T$ and $\lim_{i\to\infty} t_i = \infty$, such that

$$\eta \le h(t_i, \boldsymbol{x}(t_i)) < \rho \tag{3.100}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is a solution of system (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon_5$. Then from (3.97), (3.99) and (3.100) we have

$$\underbrace{\beta(\eta) \le \beta(h(t_i,}_{(3.100)}\underbrace{\boldsymbol{x}(t_i))) \le V(t_i,}_{(3.86)}\underbrace{\boldsymbol{x}(t_i)) \le w_{\max}(t_i,}_{(3.99)}\underbrace{t_0, w_0) < \beta(\eta)}_{(3.87)} \tag{3.101}$$

which is a contradiction. Therefore, the trivial solution of system (3.1) is (h_0, h)-asymptotically stable.

$\blacksquare$

Theorem 3.3.2. *Suppose that*

1. $h_0 \in \mathcal{H}$, $h \in \mathcal{H}$ *and* h_0 *is finer than* h;
2. $V(t, \boldsymbol{x}) \in \mathcal{V}_0$ *is* h*-positive definite, weakly* h_0*-decrescent and*

$$D^+V(t, \boldsymbol{x}) \leq -\gamma(t)\zeta(V(t, \boldsymbol{x})), \quad t \neq \tau_k, \quad (t, \boldsymbol{x}) \in \mathcal{S}_\rho(h) \qquad (3.102)$$

where $\zeta \in \mathcal{K}$, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is measurable and for any* $\rho \in \mathbb{R}_+$ *we have*

$$\lim_{t\to\infty} \int_\rho^t \gamma(s)ds = \infty; \qquad (3.103)$$

3. $V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi(V(\tau_k, \boldsymbol{x}))$ *where* $\psi \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi(w) > 0$ *for* $w > 0$ *and* $\psi(0) = 0$;
4. *there is a* $\varpi > 0$ *such that for* $w \in (0, \varpi)$ *we have*

$$-\int_{\tau_{k-1}}^{\tau_k} \gamma(s)ds + \int_w^{\psi(w)} \frac{1}{\zeta(s)}ds \leq -\nu_k, \quad \sum_{k=1}^\infty \nu_k = \infty; \qquad (3.104)$$

5. *there is a* $\rho_0 \in (0, \rho)$ *such that* $h(\tau_k, \boldsymbol{x}) < \rho_0$ *implies* $h(\tau_k^+, \boldsymbol{x}+U(k, \boldsymbol{x})) < \rho$.

Then the trivial solution of (3.1) is (h_0, h)*-asymptotically stable.* $\boxtimes$

Proof. First, let us prove the (h_0, h)-stability. From the assumption that $V(t, \boldsymbol{x})$ is h-positive definite and weakly h_0-decrescent we know that there is a $\rho_2 \in (0, \rho]$ and $\beta \in \mathcal{K}$ such that

$$h(t, \boldsymbol{x}) < \rho_2 \Rightarrow \beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}) \qquad (3.105)$$

and there are $\alpha \in \mathcal{PC}^+\mathcal{K}$ and $\varepsilon_5 > 0$ such that

$$h_0(t, \boldsymbol{x}) < \varepsilon_5 \Rightarrow V(t, \boldsymbol{x}) \leq \alpha(t, h_0(t, \boldsymbol{x})). \qquad (3.106)$$

From the fact that h_0 is finer than h we know that there is a $\varepsilon_1 > 0$ and $\kappa \in \mathcal{K}$ such that $\kappa(\varepsilon_1) < \rho_2$ and

$$h_0(t, \boldsymbol{x}) < \varepsilon_1 \Rightarrow h(t, \boldsymbol{x}) \leq \kappa(h_0(t, \boldsymbol{x})). \qquad (3.107)$$

Let $0 < \eta < \min(\rho_0, \rho_2)$, $t_0 \in \mathbb{R}_+$ and $\xi = \min(\beta(\eta), \varpi)$ be given, then since $\psi(w)$ is continuous at $w = 0$ we know that there is a constant $\theta \in (0, \xi)$ such that

$$\psi(w) < \xi \text{ for } w \in [0, \theta). \qquad (3.108)$$

Since $\alpha \in \mathcal{PC}^+\mathcal{K}$, we know that there is a $\varepsilon_2 = \varepsilon_2(t_0, \eta) > 0$ such that

$$\alpha(t_0, \varepsilon_2) < \theta. \tag{3.109}$$

Let $\varepsilon = \min(\varepsilon_5, \varepsilon_1, \varepsilon_2)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$. Then from (3.105), (3.106), (3.107) and (3.109) we know that $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon \Rightarrow$

$$\underbrace{\beta(h(t_0^+, \boldsymbol{x}_0)) \le V(t_0^+,}_{\text{from } [(3.107)\Rightarrow(3.105)]}\underbrace{\boldsymbol{x}_0) \le \alpha(t_0^+,}_{(3.106)}\underbrace{h_0(t_0^+, \boldsymbol{x}_0)) < \theta}_{(3.109)} \tag{3.110}$$

from which we have $h(t_0^+, \boldsymbol{x}_0) < \eta$. We then have the following claim:
Claim 3.3.2a: Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0\boldsymbol{x}_0)$ be a solution of system (3.1), then

$$h(t, \boldsymbol{x}) < \eta, \quad t \ge t_0. \tag{3.111}$$

If Claim 3.3.2a is false, then there is a solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$ of system (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$ and exists a $t_1 \in (\tau_k, \tau_{k+1}]$ for some k such that

$$h(t_1, \boldsymbol{x}_1(t_1)) \ge \eta \text{ and } h(t, \boldsymbol{x}_1) < \eta \text{ for } t \in [t_0, \tau_k]. \tag{3.112}$$

Since $0 < \eta < \min(\rho_0, \rho_2)$, we have $0 < \eta < \rho_0$, which leads to (in view of (3.112)) $h(t, \boldsymbol{x}_1(t)) < \rho_0$, then from condition 5 we have

$$h(t_k^+, \boldsymbol{x}_1(\tau_k) + U(k, \boldsymbol{x}_1(\tau_k))) < \rho. \tag{3.113}$$

Therefore there exists a t_2 such that

$$\eta \le h(t_2, \boldsymbol{x}_1(t_2)) < \rho \text{ and } h(t, \boldsymbol{x}_1(t)) < \rho \text{ for } t \in [t_0, t_2]. \tag{3.114}$$

From conditions 2 and 3 we have for $t \in [t_0, t_2]$

$$D^+V(t, \boldsymbol{x}_1(t)) \le -\gamma(t)\zeta(V(t, \boldsymbol{x}_1(t))), \quad t \ne \tau_i, \tag{3.115}$$

$$V(\tau_i^+, \boldsymbol{x}_1(\tau_i) + U(i, \boldsymbol{x}_1(\tau_i))) \le \psi(V(\tau_i, \boldsymbol{x}_1(\tau_i))) \tag{3.116}$$

$$i = 1, 2, \cdots, k.$$

From (3.115) we know that $V(t, \boldsymbol{x}_1(t))$ is nonincreasing in $(\tau_{i-1}, \tau_i]$ and

$$\underbrace{V(\tau_1, \boldsymbol{x}_1(\tau_1)) \le V(t_0^+,}_{\text{nonincreasing in } (\mathrm{t}_0, \tau_1]}\underbrace{\boldsymbol{x}_0) \le \alpha(t_0^+,}_{(3.106)}\ \underbrace{\varepsilon) <}_{(3.109)}\ \underbrace{\theta < \beta(\eta)}_{\theta\in(0,\xi)}. \tag{3.117}$$

Then from (3.108), (3.116) and (3.117) we have

$$\underbrace{V(\tau_1^+, \boldsymbol{x}_1(\tau_1) + U(1, \boldsymbol{x}_1(\tau_1))) <}_{(3.116),(3.108)}\ \underbrace{\xi \le \beta(\eta)}_{\xi=\min(\beta(\eta),\varpi)} \quad \text{and}$$

$$\underbrace{V(t, \boldsymbol{x}_1(t)) < \xi}_{\text{nonincreasing in } (\tau_1, \tau_2]} \quad \le \beta(\eta) \text{ for } t \in (\tau_1, \tau_2]. \tag{3.118}$$

From (3.118) and (3.115) we have

$$V(\tau_2, \boldsymbol{x}_1(\tau_2)) \leq V(\tau_1^+, \boldsymbol{x}_1(\tau_1) + U(1, \boldsymbol{x}_1(\tau_1))) < \beta(\eta) \tag{3.119}$$

from which we have the following contradiction:

$$\underbrace{\beta(\eta) \leq \beta(h(t_2,}_{(3.114)}\underbrace{\boldsymbol{x}_1(t_2))) \leq V(t_2,}_{(3.105)}\underbrace{\boldsymbol{x}_1(t_2)) < \beta(\eta)}_{(3.119)}. \tag{3.120}$$

Therefore Claim 3.3.2a is true and the trivial solution of (3.1) is (h_0, h)-stable.

We are now ready to prove the (h_0, h)-attractive property. The following conclusions are useful for proving attractive properties. Let us assume that

$$\begin{aligned} &\underbrace{V(\tau_i, \boldsymbol{x}(\tau_i)) \leq V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}) + U(i-1, \boldsymbol{x}(\tau_{i-1})))}_{(3.115)\Rightarrow\text{nonincreasing when } t\neq\tau_k} \leq \cdots \\ &\leq V(\tau_1^+, \boldsymbol{x}(\tau_1) + U(1, \boldsymbol{x}(\tau_1))) \leq V(t_0, \boldsymbol{x}_0) < \xi \leq \beta(\eta). \end{aligned} \tag{3.121}$$

Then from (3.115) and (3.116) we have

$$\int_{V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}) + U(i-1, \boldsymbol{x}(\tau_{i-1})))}^{V(\tau_i, \boldsymbol{x}(\tau_i))} \frac{1}{\zeta(s)} ds \leq -\int_{\tau_{i-1}}^{\tau_i} \gamma(s) ds \tag{3.122}$$

and

$$\int_{V(\tau_i, \boldsymbol{x}(\tau_i))}^{V(\tau_i^+, \boldsymbol{x}(\tau_i) + U(i, \boldsymbol{x}(\tau_i)))} \frac{1}{\zeta(s)} ds \leq \int_{V(\tau_i, \boldsymbol{x}(\tau_i))}^{\psi(V(\tau_i, \boldsymbol{x}(\tau_i)))} \frac{1}{\zeta(s)} ds \tag{3.123}$$

from which, in view of condition 4 and (3.121), we have

$$\begin{aligned} &\int_{V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}) + U(i-1, \boldsymbol{x}(\tau_{i-1})))}^{V(\tau_i^+, \boldsymbol{x}(\tau_i) + U(i, \boldsymbol{x}(\tau_i)))} \frac{1}{\zeta(s)} ds \\ &\leq -\int_{\tau_{i-1}}^{\tau_i} \gamma(s) ds + \int_{V(\tau_i, \boldsymbol{x}(\tau_i))}^{\psi(V(\tau_i, \boldsymbol{x}(\tau_i)))} \frac{1}{\zeta(s)} ds \leq -\nu_k. \end{aligned} \tag{3.124}$$

Because $\zeta(s) > 0$ for $s > 0$ and in view of (3.124) we have $V(\tau_i^+, \boldsymbol{x}(\tau_i) + U(i, \boldsymbol{x}(\tau_i))) \leq V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}) + U(i-1, \boldsymbol{x}(\tau_{i-1})))$. Then by using mathematical induction we have

$$\begin{aligned} &V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \leq V(\tau_{k-1}^+, \boldsymbol{x}(\tau_{k-1}) + U(k-1, \boldsymbol{x}(\tau_{k-1}))) \leq \cdots \\ &\leq V(\tau_1^+, \boldsymbol{x}(\tau_1) + U(1, \boldsymbol{x}(\tau_1))) \leq V(t_0, \boldsymbol{x}_0) < \xi \leq \beta(\eta). \end{aligned} \tag{3.125}$$

Since the trivial solution of (3.1) is (h_0, h)-stable, we can set $\eta = \min(\rho_0, \rho_2, \beta^{-1}(\varpi))$ such that $\varepsilon_0 = \varepsilon_5(t_0, \eta)$ and for each solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon_0$

$$h(t, \boldsymbol{x}(t)) < \eta, \quad t \geq t_0. \tag{3.126}$$

We then have the following claim:
Claim 3.3.2b:

$$\lim_{k\to\infty} V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) = 0. \tag{3.127}$$

If Claim 3.3.2b is not true then there is a $\vartheta > 0$ and a j such that $V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \geq \vartheta$ for $k \geq j$. Then we have for $k \geq j$

$$\zeta(\vartheta) \leq \zeta(V(\tau_k^+, \boldsymbol{x}(\tau_k)+U(k, \boldsymbol{x}(\tau_k)))) \leq \zeta(V(\tau_{k-1}^+, \boldsymbol{x}(\tau_{k-1})+U(k-1, \boldsymbol{x}(\tau_{k-1})))).$$

And then from (3.124) and condition 4 we have

$$\begin{aligned}\nu_k &\leq \int_{V(\tau_k^+, \boldsymbol{x}(\tau_k)+U(k,\boldsymbol{x}(\tau_k)))}^{V(\tau_{k-1}^+, \boldsymbol{x}(\tau_{k-1})+U(k-1,\boldsymbol{x}(\tau_{k-1})))} \frac{1}{\zeta(s)} ds \qquad \Leftarrow (3.124) \\ &\leq \frac{V(\tau_{k-1}^+, \boldsymbol{x}(\tau_{k-1}) + U(k-1, \boldsymbol{x}(\tau_{k-1}))) - V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k)))}{\zeta(\vartheta)}\end{aligned} \tag{3.128}$$

from which we have

$$\begin{aligned}&V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \\ &\leq V(\tau_{k-1}^+, \boldsymbol{x}(\tau_{k-1}) + U(k-1, \boldsymbol{x}(\tau_{k-1}))) - \nu_k \zeta(\vartheta).\end{aligned} \tag{3.129}$$

By repeating the above inequality and for a given $M \in \mathbb{N}$, we have

$$\begin{aligned}&V(\tau_{j+M}^+, \boldsymbol{x}(\tau_{j+M}) + U(j+M, \boldsymbol{x}(\tau_{j+M}))) \\ &\leq V(\tau_j^+, \boldsymbol{x}(\tau_j) + U(j, \boldsymbol{x}(\tau_j))) - \zeta(\vartheta) \sum_{l=j+1}^{j+M} \nu_l.\end{aligned} \tag{3.130}$$

Therefore we have the following contradiction:

$$\lim_{M\to\infty} V(\tau_{j+M}^+, \boldsymbol{x}(\tau_{j+M}) + U(j+M, \boldsymbol{x}(\tau_{j+M}))) = -\infty.$$

Hence, Claim 3.3.2b is true. Then we know that for a given

$$\eta \in (0, \min(\rho_0, \rho_2, \beta^{-1}(\varpi)))$$

there is such an $m > 0$ that

$$V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) < \beta(\eta) \text{ for } k \geq m. \tag{3.131}$$

Let us choose $T = \tau_m - t_0$, then from (3.105), (3.115) and (3.125) we have, for $t \geq t_0 + T$,

$$\underbrace{\beta(h(t, \boldsymbol{x}(t)) \leq V(t,}_{(3.105)}\ \underbrace{\boldsymbol{x}(t)) \leq V(\tau_k^+,}_{(3.115) \text{ and } (3.125)}\ \underbrace{\boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) < \beta(\eta)}_{(3.125)}$$

which yields because of $\beta \in \mathcal{K}$

$$h(t, \boldsymbol{x}(t)) < \eta, \quad t \geq t_0 + T.$$

Therefore the trivial solution of (3.1) is (h_0, h)-asymptotically stable. ∎

Theorem 3.3.3. *Suppose that*

1. $h_0 \in \mathcal{H}$, $h \in \mathcal{H}$ *and* h_0 *is finer than* h;
2. $V_1(t, \boldsymbol{x}) \in \mathcal{V}_0$ *is* h*-positive definite, weakly* h_0*-decrescent and*

$$\begin{aligned} D^+V_1(t, \boldsymbol{x}) \leq \chi(t)\zeta(V_1(t, \boldsymbol{x})), \quad t \neq \tau_k, \quad (t, \boldsymbol{x}) \in \mathcal{S}_\rho(h), \\ V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V_1(\tau_k, \boldsymbol{x})) \end{aligned} \tag{3.132}$$

where $\zeta \in \mathcal{K}$, $\chi \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi_k \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi_k(w) > 0$ *for* $w > 0$ *and* $\psi_k(0) = 0$;
3. *there is a* $\varpi > 0$ *such that for* $w \in (0, \varpi)$ *we have*

$$\int_{\tau_k}^{\tau_{k+1}} \chi(s)ds + \int_w^{\psi_k(w)} \frac{1}{\zeta(s)}ds \leq 0; \tag{3.133}$$

4. $V_2 \in \mathcal{V}_0$ *and*

$$\begin{aligned} V_2(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_2(\tau_k, \boldsymbol{x}), \\ D^+V_1(t, \boldsymbol{x}) + D^+V_2(t, \boldsymbol{x}) \leq -\gamma(t)\zeta(V_3(t, \boldsymbol{x})) \end{aligned} \tag{3.134}$$

where $\zeta \in \mathcal{K}$, $V_3 \in \mathcal{V}_0$ *and* $\gamma(t)$ *is integrally positive;*
5. $V_3(t, \boldsymbol{x})$ *is* h*-positive definite on* $\mathcal{S}_\rho(h)$ *and for any function* $\boldsymbol{p} : \mathbb{R}_+ \to \mathbb{R}^n$, *which is continuous on* $(\tau_k, \tau_{k+1}]$ *and*

$$\lim_{t \to \tau_k^+} \boldsymbol{p}(t) = \boldsymbol{p}(\tau_k^+)$$

exists, the function

$$\int_0^t [D^+V_3(s, \boldsymbol{p}(s)]_+ ds, (\text{ resp. } \int_0^t [D^+V_3(s, \boldsymbol{p}(s)]_- ds) \tag{3.135}$$

is uniformly continuous on $\mathbb{R}_+$, *where* $[\cdot]_+$ *(resp.* $[\cdot]_-$*) denotes that the positive (resp. negative) part is considered for all* $s \in \mathbb{R}_+$;
6. $V_3(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_3(\tau_k, \boldsymbol{x})$ *(resp.* $V_3(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq V_3(\tau_k, \boldsymbol{x})$*)*;
7. *there is a* $\rho_0 \in (0, \rho)$ *such that* $h(\tau_k, \boldsymbol{x}) < \rho_0$ *implies* $h(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) < \rho$.

Then the trivial solution of (3.1) is (h_0, h)*-asymptotically stable.* ⊠

Proof. First, let us prove the (h_0, h)-stability. From the assumption that $V_1(t, \boldsymbol{x})$ is h-positive definite and weakly h_0-decrescent we know that there is a $\rho_2 \in (0, \rho]$ and $\beta \in \mathcal{K}$ such that

$$h(t,\boldsymbol{x}) < \rho_2 \Rightarrow \beta(h(t,\boldsymbol{x})) \le V_1(t,\boldsymbol{x}) \tag{3.136}$$

and there are $\alpha \in \mathcal{PC}^+\mathcal{K}$ and $\varepsilon_5 > 0$ such that

$$h_0(t,\boldsymbol{x}) < \varepsilon_5 \Rightarrow V_1(t,\boldsymbol{x}) \le \alpha(t, h_0(t,\boldsymbol{x})). \tag{3.137}$$

From the fact that h_0 is finer than h we know that there is a $\varepsilon_1 > 0$ and $\kappa \in \mathcal{K}$ such that $\kappa(\varepsilon_1) < \rho_2$ and

$$h_0(t,\boldsymbol{x}) < \varepsilon_1 \Rightarrow h(t,\boldsymbol{x}) \le \kappa(h_0(t,\boldsymbol{x})). \tag{3.138}$$

We assume that $t_0 \in [\tau_1, \tau_2]$ and let $0 < \eta < \min(\rho_0, \rho_2)$ be given. Let us choose $\xi = \min(\beta(\eta), \varpi)$ and θ such that $\theta \in (0, \min(\xi, \psi_1(\xi)))$.

Since $\alpha \in \mathcal{PC}^+\mathcal{K}$, we know that there is a $\varepsilon_2 = \varepsilon_2(t_0, \eta) > 0$ such that

$$\alpha(t_0, \varepsilon_2) < \theta. \tag{3.139}$$

Let $\varepsilon = \min(\varepsilon_5, \varepsilon_1, \varepsilon_2)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$. Then from (3.136), (3.137), (3.138) and (3.139) we know that $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon \Rightarrow$

$$\underbrace{\beta(h(t_0^+,\boldsymbol{x}_0)) \le V_1(t_0^+,}_{\text{from } [(3.138)\Rightarrow(3.136)]}\underbrace{\boldsymbol{x}_0) \le \alpha(t_0^+,}_{(3.137)}\underbrace{h_0(t_0^+,\boldsymbol{x}_0)) < \theta}_{(3.139)} \tag{3.140}$$

from which we have $h(t_0^+, \boldsymbol{x}_0) < \eta$. We then have the following claim:
Claim 3.3.3a: Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0\boldsymbol{x}_0)$ be a solution of system (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$, then

$$h(t,\boldsymbol{x}) < \eta, \quad t \ge t_0. \tag{3.141}$$

If Claim 3.3.3a is false, then there is a solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$ of system (3.1) with $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon$ and exists a $t_1 \in (\tau_k, \tau_{k+1}]$ for some k such that

$$h(t_1, \boldsymbol{x}_1(t_1)) \ge \eta \text{ and } h(t, \boldsymbol{x}_1) < \eta \text{ for } t \in [t_0, \tau_k]. \tag{3.142}$$

Since $0 < \eta < \min(\rho_0, \rho_2)$, we have $0 < \eta < \rho_0$, which leads to (in view of (3.142)) $h(t, \boldsymbol{x}_1(t)) < \rho_0$, then from condition 7 we have

$$h(t_k^+, \boldsymbol{x}_1(\tau_k) + U(k, \boldsymbol{x}_1(\tau_k))) < \rho. \tag{3.143}$$

Therefore there exists a t_2 such that

$$\eta \le h(t_2, \boldsymbol{x}_1(t_2)) < \rho \text{ and } h(t, \boldsymbol{x}_1(t)) < \rho \text{ for } t \in [t_0, t_2]. \tag{3.144}$$

From condition 2 we have for $t \in [t_0, t_2]$

$$D^+V_1(t, \boldsymbol{x}_1(t)) \le \chi(t)\zeta(V_1(t, \boldsymbol{x}_1(t))), \quad t \ne \tau_i, \tag{3.145}$$

$$V_1(\tau_i^+, \boldsymbol{x}_1(\tau_i) + U(i, \boldsymbol{x}_1(\tau_i))) \le \psi(V_1(\tau_i, \boldsymbol{x}_1(\tau_i))), \quad i = 1, 2, \cdots, k. \tag{3.146}$$

From (3.136), (3.144) and the choice $\xi = \min(\beta(\eta), \varpi)$ we have

$$\underbrace{V_1(t, \boldsymbol{x}_1) \geq \beta(h(t, \boldsymbol{x}_1))}_{(3.136)} \underbrace{\geq}_{(3.144)} \underbrace{\beta(\eta) \geq \xi}_{\xi=\min(\beta(\eta),\varpi)} \tag{3.147}$$

Next we use mathematical induction to prove that Claim 3.3.3a is false leads to contradiction. In the first step, let us suppose $t_2 \in (t_0, \tau_2]$, then we have

$$\begin{aligned} &\underbrace{\int_{\psi_1(\xi)}^{\xi} \frac{1}{\zeta(s)} ds < \int_{\theta}^{\xi}}_{\theta\in(0,\min(\xi,\psi_1(\xi)))} \underbrace{\frac{1}{\zeta(s)} ds < \int_{V_1(t_0^+,\boldsymbol{x}_0)}^{\xi}}_{(3.140)} \underbrace{\frac{1}{\zeta(s)} ds \leq \int_{V_1(t_0^+,\boldsymbol{x}_0)}^{V_1(t_2,\boldsymbol{x}_1(t_2))}}_{(3.147)} \frac{1}{\zeta(s)} ds \\ &\leq \int_{t_0}^{t_2} \underbrace{\chi(s) ds \leq \int_{\tau_1}^{\tau_2}}_{t_0\in[\tau_1,\tau_2]} \chi(s) ds \end{aligned} \tag{3.148}$$

from which we have

$$\int_{\tau_1}^{\tau_2} \chi(s) ds + \int_{\xi}^{\psi_1(\xi)} \frac{1}{\zeta(s)} ds > 0 \tag{3.149}$$

which results in a contradiction to condition 3.

In the second step, let us consider the case when $t_2 \in (\tau_i, \tau_{i+1}]$. Let us suppose for $t \in [t_0, \tau_i]$, $V_1(t, \boldsymbol{x}_1(t)) < \xi$, then from (3.145) we have for $t \in (\tau_i, \tau_{i+1}]$

$$\int_{V_1(\tau_1^+,\boldsymbol{x}_1(\tau_1)+U(1,\boldsymbol{x}_1(\tau_1)))}^{V_1(t,\boldsymbol{x}_1(t))} \frac{1}{\zeta(s)} ds \leq \int_{\tau_i}^{t} \chi(s) ds \leq \int_{\tau_i}^{\tau_{i+1}} \chi(s) ds. \tag{3.150}$$

Since $V_1(\tau_i^+, \boldsymbol{x}_1(\tau_i) + U(i, \boldsymbol{x}_1(\tau_i))) \leq \psi_i(V_1(\tau_i, \boldsymbol{x}_1(\tau_i)))$, then we have

$$\int_{V_1(\tau_i,\boldsymbol{x}_1(\tau_i))}^{V_1(\tau_i^+,\boldsymbol{x}_1(\tau_i)+U(i,\boldsymbol{x}_1(\tau_i)))} \frac{1}{\zeta(s)} ds \leq \int_{V_1(\tau_i,\boldsymbol{x}_1(\tau_i))}^{\psi_i(V_1(\tau_i,\boldsymbol{x}_1(\tau_i)))} \frac{1}{\zeta(s)} ds \tag{3.151}$$

from which and (3.150) we know that, for $t \in (\tau_i, \tau_{i+1}]$,

$$\begin{aligned} \int_{V_1(\tau_i,\boldsymbol{x}_1(\tau_i))}^{V_1(t,\boldsymbol{x}_1(t))} \frac{1}{\zeta(s)} ds &\leq \int_{\tau_i}^{\tau_{i+i}} \chi(s) ds \\ &\quad + \int_{V_1(\tau_i,\boldsymbol{x}_1(\tau_i))}^{\psi_i(V_1(\tau_i,\boldsymbol{x}_1(\tau_i)))} \frac{1}{\zeta(s)} ds \leq 0. \end{aligned} \tag{3.152}$$

Since $\zeta(s) > 0$ for $s > 0$, from above we have

$$V_1(t, \boldsymbol{x}_1(t)) \leq V_1(\tau_i, \boldsymbol{x}_1(\tau_i)) < \xi \text{ for } t \in (\tau_i, \tau_{i+1}].$$

Then, as the conclusion of our mathematical induction, we know that $V_1(t, \boldsymbol{x}_1(t)) < \xi$ for $t \in [t_0, t_2]$. Then by (3.136), (3.144) and $\xi = \min(\beta(\eta), \varpi)$ we have the following contradiction:

$$\underbrace{\beta(\eta) \leq \beta(h(t_2,}_{(3.144)} \underbrace{\boldsymbol{x}_1(t_2))) \leq V_1(t_2,}_{(3.136)} \underbrace{\boldsymbol{x}_1(t_2)) < \beta(\eta)}_{\xi=\min(\beta(\eta),\varpi)}.$$

Therefore, Claim 3.3.3a is true and the trivial solution of (3.1) is (h_0, h)-stable.

We are now ready to prove the (h_0, h)-attractive property. Since the trivial solution of (3.1) is (h_0, h)-stable, we can set $\eta = \rho_0$ such that $\varepsilon_0 = \varepsilon_5(t_0, \rho_0)$ and $h_0(t_0^+, \boldsymbol{x}_0) < \varepsilon_0$ imply, for each solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (3.1), that

$$h(t, \boldsymbol{x}(t)) < \varepsilon_5, \quad t \geq t_0. \tag{3.153}$$

We then have the following claim:
Claim 3.3.3b:

$$\lim_{t \to \infty} \inf V_3(t, \boldsymbol{x}(t)) = 0. \tag{3.154}$$

If Claim 3.3.3b is not true then for some $T > 0$, there is an $a > 0$ such that

$$V_3(t, \boldsymbol{x}(t)) \geq a, \quad t \geq t_0 + T. \tag{3.155}$$

We can choose a sequence

$$t_0 + T < a_1 < b_1 < \cdots < a_i < b_i < \cdots$$

such that $b_i - a_i \geq a$. Since $\chi \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\zeta \in \mathcal{K}$, $\tau_{k+1} > \tau_k$ and $w > 0$, it follows from (3.133) that $\psi_k(w) \leq w$, then from condition 4 and (3.155) we have the following contradiction:

$$\begin{aligned}
&\lim_{t \to \infty} V_1(t, \boldsymbol{x}(t)) + V_2(t, \boldsymbol{x}(t)) \\
&\leq V_1(t_0^+, \boldsymbol{x}_0) + V_2(t_0^+, \boldsymbol{x}_0) - \int_{t_0}^{\infty} \gamma(s)\zeta(V_3(s, \boldsymbol{x}(s)))ds \\
&\leq V_1(t_0^+, \boldsymbol{x}_0) + V_2(t_0^+, \boldsymbol{x}_0) - \zeta(a) \int_{\bigcup_{i=1}^{\infty}[a_i, b_i]} \gamma(s)ds = -\infty.
\end{aligned} \tag{3.156}$$

Therefore Claim 3.3.3b is true. Let us assume that

$$\lim_{t \to \infty} \sup V_3(t, \boldsymbol{x}(t)) > 0,$$

then there is a $\nu > 0$ such that

$$\lim_{t \to \infty} \sup V_3(t, \boldsymbol{x}(t)) > 2\nu.$$

Let us only consider the case when condition 5 holds with $[\cdot]_+$ and condition 6 holds with $V_3(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \le V_3(\tau_k, \boldsymbol{x}(\tau_k))$.

Since (3.154) is true, we can find a sequence

$$t_0 < t_1^a < t_1^b < \cdots < t_i^a < t_i^b < \cdots$$

such that

$$V_3(t_i^a, \boldsymbol{x}(t_i^a)) = \nu \text{ and } V_3(t_i^b, \boldsymbol{x}(t_i^b)) = 2\nu, \quad i = 1, 2, \cdots \tag{3.157}$$

from which and condition 6, we know there is a sequence

$$t_0 < a_1 < b_1 < \cdots < a_i < b_i < \cdots$$

such that $t_i^a \le a_i < b_i \le t_i^b$ and

$$\begin{aligned} &V_3(a_i, \boldsymbol{x}(a_i)) = \nu, V_3(b_i, \boldsymbol{x}(b_i)) = 2\nu \text{ and} \\ &V_3(t, \boldsymbol{x}(t)) \in [\nu, 2\nu] \text{ for } t \in [a_i, b_i]. \end{aligned} \tag{3.158}$$

However, we have

$$\begin{aligned} 0 < \nu &= V_3(b_i, \boldsymbol{x}(b_i)) - V_3(a_i, \boldsymbol{x}(a_i)) \\ &\le \int_{a_i}^{b_i} [D^+ V_3(s, \boldsymbol{x}(s))]_+ ds, \quad i = 1, 2, \cdots \end{aligned} \tag{3.159}$$

from which and condition 5 we have, for some $\varsigma > 0$

$$b_i - a_i \ge \varsigma, \quad i = 1, 2, \cdots . \tag{3.160}$$

Therefore, from (3.158), (3.160) and condition 4 we have the following contradiction:

$$\begin{aligned} &\lim_{t\to\infty} V_1(t, \boldsymbol{x}(t)) + V_2(t, \boldsymbol{x}(t)) \\ &\le V_1(t_0^+, \boldsymbol{x}_0) + V_2(t_0^+, \boldsymbol{x}_0) - \int_{t_0}^{\infty} \gamma(s)\zeta(V_3(s, \boldsymbol{x}(s)))ds \qquad \Leftarrow \text{(condition 4)} \\ &\le V_1(t_0^+, \boldsymbol{x}_0) + V_2(t_0^+, \boldsymbol{x}_0) - \zeta(\nu) \int_{\bigcup_{i=1}^{\infty}[a_i, b_i]} \gamma(s)ds = -\infty. \end{aligned} \tag{3.161}$$

(3.154) and the above contradiction imply that $\lim_{t\to\infty} V_3(t, \boldsymbol{x}(t)) = 0$. Then it follows form $V_3(t, \boldsymbol{x}(t))$ being h-positive that $\lim_{t\to\infty} h(t, \boldsymbol{x}(t)) = 0$. Therefore the trivial solution of (3.1) is (h_0, h)-asymptotically stable. ■

3.4 Multicomparison Systems

In many cases we study impulsive control problems by using more than one comparison system. We called this kind of method as stability based on multicomparison systems. In this case, vector Lyapunov functions should be used. The purpose of using more than one comparison system is to explore the flexibility provided by different Lyapunov functions.

Definition 3.4.1. *Multicomparison system*
Let $\boldsymbol{V} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+^m$ *satisfy* $V_i \in \mathcal{V}_0$, $i = 1, 2, \cdots, m$, *and*

$$\begin{aligned} D^+\boldsymbol{V}(t, \boldsymbol{x}) &\preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x})), \quad t \neq \tau_k, \\ \boldsymbol{V}(t, \boldsymbol{x} + U(k, \boldsymbol{x})) &\preceq \boldsymbol{\psi}_k(\boldsymbol{V}(t, \boldsymbol{x})), \quad t = \tau_k \end{aligned} \tag{3.162}$$

where $\boldsymbol{g} : \mathbb{R}_+ \times \mathbb{R}_+^m \to \mathbb{R}_+^m$ *is continuous in* $(\tau_{k-1}, \tau_k] \times \mathbb{R}^m$ *and for each* $\boldsymbol{p} \in \mathbb{R}^m$, $k = 1, 2, \cdots$, *the limit*

$$\lim_{(t,\boldsymbol{q}) \to (\tau_k^+, \boldsymbol{p})} \boldsymbol{g}(t, \boldsymbol{q}) = \boldsymbol{g}(\tau_k^+, \boldsymbol{p})$$

exists. $\boldsymbol{g}(t, \boldsymbol{q})$ *is quasimonotone nondecreasing in* $\boldsymbol{q}$ *and* $\boldsymbol{\psi}_k : \mathbb{R}_+^m \to \mathbb{R}_+^m$ *is nondecreasing. Then the following system*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}), \quad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)), \\ \boldsymbol{w}(t_0^+) &= \boldsymbol{w}_0 \geq 0 \end{aligned} \tag{3.163}$$

is the multicomparison system of (3.1). ⊠

Theorem 3.4.1. *Assume that*

1. $\{\tau_k\}$ *satisfies* $0 \leq t_0 < \tau_1 < \tau_2 < \cdots$ *and* $\lim_{k\to\infty} \tau_k = \infty$;
2. $\boldsymbol{p} \in \mathcal{PC}^1[\mathbb{R}_+, \mathbb{R}^n]$ *and* $\boldsymbol{p}(t)$ *is left-continuous at* τ_k, $k = 1, 2, \cdots$;
3. $\boldsymbol{g} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\boldsymbol{g}(t, \boldsymbol{w})$ *is quasimonotone nondecreasing in* $\boldsymbol{w}$ *for every* t. *And for* $k = 1, 2, \cdots$, $\boldsymbol{\psi}_k \in C[\mathbb{R}^n, \mathbb{R}^n]$ *and* $\boldsymbol{\psi}_k(\boldsymbol{w})$ *is nondecreasing in* $\boldsymbol{w}$ *and*

$$\begin{aligned} &D_-\boldsymbol{p}(t) \preceq \boldsymbol{g}(t, \boldsymbol{p}(t)), \quad t \neq \tau_k, \quad \boldsymbol{p}(t_0) \preceq \boldsymbol{w}_0, \\ &\boldsymbol{p}(\tau_k^+) \preceq \boldsymbol{\psi}_k(\boldsymbol{p}(\tau_k)); \end{aligned} \tag{3.164}$$

4. $\boldsymbol{w}_{\max}(t)$ *is the maximal solution of the following impulsive differential equation on* $[t_0, \infty)$:

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}), \quad t \neq \tau_k, \quad \boldsymbol{w}(t_0) = \boldsymbol{w}_0, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)). \end{aligned} \tag{3.165}$$

Then

$$\boldsymbol{p}(t) \preceq \boldsymbol{w}_{\max}(t), \quad t \geq t_0.$$

⊠

Proof. From the classical comparison theorem we know that $\boldsymbol{p}(t) \preceq \boldsymbol{w}_{\max}(t)$ for $t \in [t_0, \tau_1]$. Since $\boldsymbol{p}(\tau_1) \preceq \boldsymbol{w}_{\max}(\tau_1)$ and $\boldsymbol{\psi}_1(\boldsymbol{w})$ is nondecreasing we have

$$\boldsymbol{p}(\tau_1^+) \preceq \boldsymbol{\psi}_1(\boldsymbol{p}(\tau_1)) \preceq \boldsymbol{\psi}_1(\boldsymbol{w}_{\max}(\tau_1)) = \boldsymbol{w}_1^+. \tag{3.166}$$

For $t \in (\tau_1, \tau_2]$ by using classical comparison theorem we know that $\boldsymbol{p}(t) \preceq \boldsymbol{w}_{\max}(t)$, where $\boldsymbol{w}_{\max}(t) = \boldsymbol{w}_{\max}(t, \tau_1, \boldsymbol{w}_1^+)$ is the maximal solution of (3.165) on $t \in [\tau_1, \tau_2]$. Similarly, we have

$$\boldsymbol{p}(\tau_2^+) \preceq \boldsymbol{\psi}_2(\boldsymbol{p}(\tau_2)) \preceq \boldsymbol{\psi}_2(\boldsymbol{w}_{\max}(\tau_2)) = \boldsymbol{w}_2^+. \tag{3.167}$$

By repeating the same process we can finish the proof. ■

Theorem 3.4.2. *Let $\boldsymbol{w}_{\max} = \boldsymbol{w}(t, t_0, \boldsymbol{w}_0)$ be the maximal solution of the multicomparison system (3.163) on $[t_0, \infty)$. Then for any solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (3.1) on $[t, \infty)$, $\boldsymbol{V}(t_0^+, \boldsymbol{x}_0) \preceq \boldsymbol{w}_0$ implies*

$$\boldsymbol{V}(t, \boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max}(t), \quad t \geq t_0.$$

⊠

Proof. Let $\tilde{\boldsymbol{V}}(t) = \boldsymbol{V}(t, \boldsymbol{x}(t))$ for $t \neq \tau_k$ such that for a small $\delta > 0$ we have

$$\begin{aligned}\tilde{\boldsymbol{V}}(t+\delta) - \tilde{\boldsymbol{V}}(t) = {} & \boldsymbol{V}(t+\delta, \boldsymbol{x}(t+\delta)) - \boldsymbol{V}(t+\delta, \boldsymbol{x}(t) + \delta(\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}))) \\ & + \boldsymbol{V}(t+\delta, \boldsymbol{x}(t) + \delta(\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}))) - \boldsymbol{V}(t, \boldsymbol{x}(t)).\end{aligned} \tag{3.168}$$

Since $\boldsymbol{V}(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ for $t \in (\tau_k, \tau_{k+1}]$, from (3.162) we have

$$\begin{aligned} D^+\tilde{\boldsymbol{V}}(t) \preceq \boldsymbol{g}(t, \tilde{\boldsymbol{V}}(t)), \quad & t \neq \tau_k, \tilde{\boldsymbol{V}}(t_0^+) \preceq \boldsymbol{w}_0, \\ \tilde{\boldsymbol{V}}(\tau_k^+) = \boldsymbol{V}(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) & \preceq \boldsymbol{\psi}_k(\tilde{\boldsymbol{V}}(\tau_k)). \end{aligned} \tag{3.169}$$

Therefore, followed Theorem 3.4.1 we finish the proof. ■

Theorem 3.4.3. *Suppose that*

1. $\boldsymbol{V} : \mathbb{R}_+ \times \mathcal{S}_\rho \to \mathbb{R}_+^m$, $V_i \in \mathcal{V}_0$, $i = 1, 2, \cdots, m$, *and*

$$D^+\boldsymbol{V}(t, \boldsymbol{x}) \preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x})), \quad t \neq \tau_k, \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho \tag{3.170}$$

   where $\boldsymbol{g}(t, 0) = 0$ and satisfies other conditions listed in Definition 3.4.1.
2. *There is a ρ_0 such that $\boldsymbol{x}_0 \in \mathcal{S}_{\rho_0}$ implies that $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_\rho$, and*

$$\boldsymbol{V}(t, \boldsymbol{x} + U(k, \boldsymbol{x})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}(t, \boldsymbol{x})), \quad t = \tau_k \tag{3.171}$$

   where $\boldsymbol{\psi}_k : \mathbb{R}_+^m \to \mathbb{R}_+^m$ satisfies other conditions listed in Definition 3.4.1.
3. *For $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$, $\beta(\|\boldsymbol{x}\|) \leq Æ(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|)$, where $\alpha, \beta \in \mathcal{K}$ and $Æ(t, \boldsymbol{x})$ can be any of the following:*

$$\begin{aligned} Æ(t, \boldsymbol{x}) &= \sum_{i=1}^m V_i(t, \boldsymbol{x}), \\ Æ(t, \boldsymbol{x}) &= \sum_{i=1}^m d_i V_i(t, \boldsymbol{x}), \quad d_i \geq 0, \\ Æ(t, \boldsymbol{x}) &= \max_{1 \leq i \leq m} V_i(t, \boldsymbol{x}). \end{aligned} \tag{3.172}$$

Then the stability properties of the trivial solution of (3.163) imply the corresponding stability properties of trivial solution of (3.1). ⊠

Proof. By using the conclusion of Theorem 3.4.2 and using the similar processes in the proof of Theorem 3.1.3 the proof can be completed. ■

Let us consider the stability of the trivial solution of the following impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \quad k \in \mathbb{N}, \quad t \in \mathbb{R}_+ \end{aligned} \qquad (3.173)}$$

where $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$, $\rho_0 > 0$. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ satisfies $\boldsymbol{f}(t, 0) = 0$ for $t \in \mathbb{R}_+$ and there is a constant $L > 0$ such that $\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ for $t \in \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_{\rho_0}$, $U \in C[\mathbb{N} \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ and $U(k, 0) = 0$. There is a constant $\rho \in (0, \rho_0)$ such that if $\boldsymbol{x} \in \mathcal{S}_\rho$, then $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_{\rho_0}$. $\tau_k \in C[\mathcal{S}_{\rho_0}, \mathbb{R}_+]$, $k \in \mathbb{N}$ satisfy

$$0 = \tau_0(\boldsymbol{x}) < \tau_1(\boldsymbol{x}) < \tau_2(\boldsymbol{x}) < \cdots, \quad \lim_{k \to \infty} \tau_k(\boldsymbol{x}) = \infty, \quad \boldsymbol{x} \in \mathcal{S}_{\rho_0}.$$

We assume that there is no beating phenomenon.

Let us define

$$\Omega \triangleq \{\boldsymbol{w} \in \mathbb{R}^m \mid \|\boldsymbol{w}\| < \varsigma\}, \quad \varsigma \in (0, \infty].$$

Let $\boldsymbol{V} \in C[\mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \mathbb{R}^m_+]$ be locally Lipschitzian and satisfy $V_i \in \mathcal{V}_0$, $i = 1, 2, \cdots, m$, and

$$\begin{aligned} D^+\boldsymbol{V}(t, \boldsymbol{x}) &\preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x})), \quad t \neq \tau_k, \quad \boldsymbol{x} \in \mathcal{S}_{\rho_0}, \\ \boldsymbol{V}(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) &\preceq \boldsymbol{\psi}_k(\boldsymbol{V}(t, \boldsymbol{x})), \quad t = \tau_k, \quad \boldsymbol{x} \in \mathcal{S}_\rho. \end{aligned} \qquad (3.174)$$

Then the *multicomparison system* of impulsive control system (3.173) is given by

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}), \quad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)), \\ \boldsymbol{w}(t_0^+) &= \boldsymbol{w}_0 \geq 0 \end{aligned} \qquad (3.175)$$

where $t \in \mathbb{R}_+$, $k \in \mathbb{N}$, $\boldsymbol{g} \in C[(\tau_k, \tau_{k+1}] \times \Omega, \mathbb{R}^m]$ is quasimonotonically increasing, $\boldsymbol{g}(t, 0) = 0$ for $t \in \mathbb{R}_+$, for $\boldsymbol{q} \in \Omega$ and $k \in \mathbb{N}$ the following limit exists:

$$\lim_{(t, \boldsymbol{w}) \to (\tau_k^+, \boldsymbol{q})} \boldsymbol{g}(t, \boldsymbol{w}) = \boldsymbol{g}(\tau_k^+, \boldsymbol{q}),$$

$\boldsymbol{\psi}_k : \Omega \to \mathbb{R}^m$ are nondecreasing and $\boldsymbol{\psi}_k(0) = 0$ for $k \in \mathbb{N}$. Let $\boldsymbol{e}_m \triangleq \mathrm{col}(1, \cdots, 1) \in \mathbb{R}^m$, then there is such a $\varpi > 0$ that

$$\varXi \triangleq \{\boldsymbol{w} \in \mathbb{R}^m | \ -\varpi \boldsymbol{e}_m \preceq \boldsymbol{w} \preceq \varpi \boldsymbol{e}_m\} \subset \varOmega$$

and $\boldsymbol{\psi}_k(\boldsymbol{w}) \in \varOmega$ for any $\boldsymbol{w} \in \varXi$. We then have the following comparison theorem.

Theorem 3.4.4. *Assume that there is such an $\alpha \in \mathcal{K}$ that for $t \in \mathbb{R}_+$ and $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$*

$$\alpha(\|\boldsymbol{x}\|) \le \max_{1\le i\le m} V_i(t,\boldsymbol{x}),$$
$$\sup_{(t,\boldsymbol{x})\in\mathbb{R}_+\times\mathcal{S}_{\rho_0}} \|V(t,\boldsymbol{x})\| < \varsigma. \tag{3.176}$$

Then, if the trivial solution of the comparison system (3.175) is stable (resp. asymptotically stable), then the trivial solution of system (3.173) is stable (resp. asymptotically stable); ⊠

Proof. Let $t_0 \in \mathbb{R}_+$, $\boldsymbol{x}(t) = \boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ with $\boldsymbol{x}(t_0,t_0,\boldsymbol{x}_0) = \boldsymbol{x}_0 \in \mathcal{S}_{\rho_0}$ be a solution of system (3.173) and $\boldsymbol{w}_{\max}(t) = \boldsymbol{w}_{\max}(t,t_0,\boldsymbol{w}_0)$ with $\boldsymbol{w}_{\max}(t_0,t_0,\boldsymbol{w}_0) = \boldsymbol{w}_0 \in \varOmega$ be the maximal solution of the comparison system (3.175). Let us choose t_0, $\boldsymbol{x}_0$ and $\boldsymbol{w}_0$ such that $\boldsymbol{V}(t_0^+,\boldsymbol{x}_0) \preceq \boldsymbol{w}_0$, then it follows from Theorem 3.4.2 that

$$\boldsymbol{V}(t,\boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max}(t), \quad t > t_0, \tag{3.177}$$

from which and the assumption on α we have

$$\alpha(\|\boldsymbol{x}(t)\|) \le \max_{1\le i\le m} V_i(t,\boldsymbol{x}(t) \le \max_{1\le i\le m} w_{\max\ i}(t). \tag{3.178}$$

Let us first prove the stability; namely, in this case the trivial solution of system (3.175) is stable. Let us choose such a $\delta \in (0,\rho)$ that $\alpha(\delta) < \varpi$, then from the stability of the trivial solution of system (3.175) we know that there is such a $\kappa = \kappa(t_0,\delta) \in (0,\varpi)$ that if $0 \preceq \boldsymbol{w}_0 \preceq \kappa\boldsymbol{e}_m$ then

$$0 \preceq \boldsymbol{w}_{\max}(t) \prec \alpha(\delta)\boldsymbol{e}_m \prec \varpi\boldsymbol{e}_m, \quad t > t_0. \tag{3.179}$$

From the assumptions on $\boldsymbol{V}(t,\boldsymbol{x}(t))$ we know that there is a $\varepsilon = \varepsilon(t_0,\delta) \in (0,\min(\alpha(\delta),\alpha(\varpi)))$ such that

$$\|\boldsymbol{x}_0\| < \varepsilon \quad \Rightarrow \quad -\kappa\boldsymbol{e}_m \preceq \boldsymbol{V}(t_0,\boldsymbol{x}_0) \preceq \kappa\boldsymbol{e}_m. \tag{3.180}$$

Let us choose $\boldsymbol{w}_0 = \kappa\boldsymbol{e}_m$ then it follows from (3.177), (3.178), (3.179) and (3.180) that if $\|\boldsymbol{x}_0\| < \varepsilon$ then for any $t > t_0$ we have $\|\boldsymbol{x}(t)\| < \delta < \rho$. This conclusion is the result of the following reasoning flow chart:

$$
\left.\begin{array}{r}
\|\boldsymbol{x}_0\| < \underbrace{\varepsilon \Rightarrow \boldsymbol{V}(t_0}_{(3.180)}, \boldsymbol{x}_0) \preceq \kappa \boldsymbol{e}_m \\
\text{choose } \boldsymbol{w}_0 = \kappa \boldsymbol{e}_m
\end{array}\right\} \Rightarrow \boldsymbol{V}(t_0, \boldsymbol{x}_0) \preceq \boldsymbol{w}_0 \Rightarrow
$$

$$
\left.\begin{array}{r}
\underbrace{\Rightarrow \boldsymbol{V}(t,}_{(3.177)} \boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max}(t) \\
(3.179)
\end{array}\right\} \Rightarrow
$$

$$
\left.\begin{array}{r}
\Rightarrow \boldsymbol{V}(t, \boldsymbol{x}(t)) \prec \alpha(\delta) \boldsymbol{e}_m \\
(3.178)
\end{array}\right\} \Rightarrow
$$

$$
\left.\begin{array}{r}
\Rightarrow \alpha(\|\boldsymbol{x}(t)\|) < \alpha(\delta) \\
\alpha \in \mathcal{K}
\end{array}\right\} \Rightarrow
$$

$$
\left.\begin{array}{r}
\Rightarrow \|\boldsymbol{x}(t)\| < \delta \\
\delta \in (0, \rho)
\end{array}\right\} \Rightarrow \|\boldsymbol{x}(t)\| < \delta < \rho. \tag{3.181}
$$

This proves that the trivial solution of system (3.173) is stable. Asymptotic stability can be proved in similar manner. ■

Theorem 3.4.5. *Assume that there are such $\alpha, \beta \in \mathcal{K}$ that for $t \in \mathbb{R}_+$ and $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$*

$$
\begin{aligned}
\alpha(\|\boldsymbol{x}\|) \le \max_{1 \le i \le m} V_i(t, \boldsymbol{x}) \le \beta(\|\boldsymbol{x}\|), \\
\sup_{(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}} \|V(t, \boldsymbol{x})\| < \varsigma.
\end{aligned} \tag{3.182}
$$

Then, if the trivial solution of the comparison system (3.175) is uniformly stable (resp. uniformly asymptotically stable), then the trivial solution of system (3.173) is uniformly stable (resp. uniformly asymptotically stable). ⊠

Proof. The proof is similar as that of Theorem 3.4.4. Let us only prove the uniform stability. It follows from Theorem 3.4.4 that the trivial solution of system (3.173) is stable. Since

$$
\max_{1 \le i \le m} V_i(t, \boldsymbol{x}) \le \beta(\|\boldsymbol{x}\|)
$$

we can choose a ε independent of t_0 such that

$$
\beta(\varepsilon) \le \kappa.
$$

Then, whenever $\|\boldsymbol{x}_0\| < \varepsilon$ we have

$$
\max_{1 \le i \le m} V_i(t_0, \boldsymbol{x}_0) \le \beta(\|\boldsymbol{x}_0\|) < \beta(\varepsilon) \le \kappa
$$

from which we get the same conclusion as in (3.180). Therefore we prove that the trivial solution of system (3.173) is uniformly stable. The uniform asymptotic stability can be proved similarly. ■

4. Impulsive Control with Fixed-time Impulses

In this chapter we study impulsive control of nonlinear systems with fixed-time impulses. In this kind of impulsive control system, the impulses are generated by an *independent* "clock signal" such as those used in digital controllers.

4.1 Lyapunov's Second Method

Let us consider the stability of the zero solution of the following impulsive differential equation:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \quad k \in \mathbb{N}, \quad t \in \mathbb{R}_+ \end{aligned} \qquad (4.1)}$$

where $\boldsymbol{x} \in \mathcal{S}_\rho$, $\rho > 0$. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ satisfies $\boldsymbol{f}(t, 0) = 0$ for $t \in \mathbb{R}_+$ and there is a constant $L > 0$ such that $\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ for $t \in \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_{\rho_0}$, $U \in C[\mathbb{N} \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ and $U(k, 0) = 0$. There is a constant $\rho \in (0, \rho_0)$ such that if $\boldsymbol{x} \in \mathcal{S}_\rho$, then $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_{\rho_0}$. $\tau_k \in C[\mathcal{S}_{\rho_0}, \mathbb{R}_+]$, $k \in \mathbb{N}$ satisfy

$$0 = \tau_0(\boldsymbol{x}) < \tau_1(\boldsymbol{x}) < \tau_2(\boldsymbol{x}) < \cdots, \quad \lim_{k \to \infty} \tau_k(\boldsymbol{x}) = \infty, \quad \boldsymbol{x} \in \mathcal{S}_{\rho_0}.$$

We assume that there is no beating phenomenon.

Definition 4.1.1. *For $(t, \boldsymbol{x}) \in \mathfrak{G}$, we define the derivative of the function $V \in \mathcal{V}_1$ with respect to system (4.1) as*

$$\dot{V}(t, \boldsymbol{x}) \triangleq \frac{\partial V(t, \boldsymbol{x})}{\partial t} + \frac{\partial V(t, \boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}). \qquad (4.2)$$

⊠

Theorem 4.1.1. *Assume that there are $V \in \mathcal{V}_1$ and $\alpha \in \mathcal{K}$ such that*

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 71−118, 2001.

$$\alpha(\|\boldsymbol{x}\|) \le V(t, \boldsymbol{x}), \quad \textit{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \tag{4.3}$$

$$\dot{V}(t, \boldsymbol{x}) \le 0, \quad \textit{for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{4.4}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(t, \boldsymbol{x}), \ \textit{for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho). \tag{4.5}$$

Then the trivial solution of system (4.1) is

1. *stable.*
2. *uniformly stable if for some* $\beta \in \mathcal{K}$

$$V(t, \boldsymbol{x}) \le \beta(\|\boldsymbol{x}\|), \quad \textit{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}. \tag{4.6}$$

⊠

Proof. We first prove conclusion 1. For a given $\delta > 0$ and $t_0 \in \mathbb{R}_+$, it follows from $V \in \mathcal{V}_1$ that there is such a $\varepsilon = \varepsilon(t_0, \delta) > 0$ that

$$\sup_{\boldsymbol{x} \in \mathcal{S}_\varepsilon} |V(t_0^+, \boldsymbol{x})| < \min(\alpha(\delta), \alpha(\rho)).$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (4.1) with $\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ such that $\boldsymbol{x}_0 \in \mathcal{S}_{\rho_0}$ and $\|\boldsymbol{x}_0\| < \varepsilon$. It follows from (4.4) and (4.5) that $V(t, \boldsymbol{x})$ is nonincreasing. Then, it follows from (4.3) we have

$$\alpha(\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\|) \le V(t, \boldsymbol{x}(t)) \le V(t_0^+, \boldsymbol{x}_0) < \min(\alpha(\delta), \alpha(\rho))$$

from which we have for all $t > t_0$

$$\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| < \min(\delta, \rho).$$

Therefore, whenever $\|\boldsymbol{x}_0\| < \varepsilon$ we have $\|\boldsymbol{x}(t)\| < \delta$ for all $t \in \mathbb{R}_+$; namely, the trivial solution of system (4.1) is stable.

If condition (4.6) holds, then we can choose a ε independent of t_0 such that

$$\beta(\varepsilon) < \min(\alpha(\delta), \alpha(\rho)).$$

This proves that the trivial solution is uniformly stable. ■

For $t \in \mathbb{R}_+$, $\chi \in \mathbb{R}_+$, $V \in \mathcal{V}_0$ and $\alpha \in \mathcal{K}$ we define the sets

$$\Theta_\chi(t, \alpha) \triangleq \{\boldsymbol{x} \in \mathcal{S}_{\rho_0} | \ V(t^+, \boldsymbol{x}) < \alpha(\chi)\}.$$

Let the set

$$\Xi(t_0) \triangleq \{\boldsymbol{x}_0 \in \mathcal{S}_{\rho_0} | \ \boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \to 0 \text{ as } t \to \infty\}$$

be the *basin of attraction!of the origin* at t_0.

Theorem 4.1.2. *Assume that $V, V_1 \in \mathcal{V}_1$ and there are $\alpha, \beta, \varrho \in \mathcal{K}$ such that*

$$\alpha(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}), \tag{4.7}$$

$$\beta(\|\boldsymbol{x}\|) \leq V_1(t, \boldsymbol{x}), \quad \textit{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \tag{4.8}$$

$$\dot{V}(t, \boldsymbol{x}) \leq -\varrho(V_1(t, \boldsymbol{x})), \quad \textit{for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{4.9}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x}), \textit{ for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho), \tag{4.10}$$

and $\dot{V}_1(t, \boldsymbol{x})$ is bounded from above (resp., from below) in $\mathfrak{G}$ and

$$V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_1(t, \boldsymbol{x}) \ (\textit{resp., } V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq V_1(t, \boldsymbol{x})),$$
$$\textit{for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho). \tag{4.11}$$

Then we have the following conclusions:

1. *If $0 < a < \rho$ and $t_0 \in \mathbb{R}_+$, then $\Xi(t_0) \supset \Theta_a(t_0, \alpha)$;*
2. *The trivial solution of system (4.1) is asymptotically stable.*

⊠

Proof. Let us choose an $a \in (0, \rho)$, then from (4.7) we have

$$\Theta_a(t, \alpha) \triangleq \{\boldsymbol{x} \in \mathcal{S}_{\rho_0} |\ V(t^+, \boldsymbol{x}) < \underbrace{\alpha(a)\} \subset \mathcal{S}_a}_{(4.7)} \underbrace{\subset \mathcal{S}_{\rho_0}}_{\rho_0 > \rho}, \quad t \in \mathbb{R}_+.$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (4.1) with $\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ such that $t_0 \in \mathbb{R}_+$ and $\boldsymbol{x}_0 \in \Theta_a(t, \alpha)$. By using the similar procedure of the proof of Theorem 4.1.1 we know that $\boldsymbol{x}(t) \in \mathcal{S}_a$ for all $t > t_0$. We then have the following claim:
Claim 4.1.2

$$\lim_{t \to \infty} \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| = 0 \text{ for any } \boldsymbol{x}_0 \in \Theta_a(t_0, \alpha). \tag{4.12}$$

Let us suppose that Claim 4.1.2 is false, then there are $\boldsymbol{x}_0 \in \Theta_a(t_0, \alpha)$, $\eta, \xi > 0$ and a sequence $\{s_k\}_{k=1}^{\infty}$ satisfying $s_{k+1} - s_k \geq \eta$ and $\|\boldsymbol{x}(s_k, t_0, \boldsymbol{x}_0)\| \geq \xi, k \in \mathbb{N}$. It follows from (4.8) that

$$|V_1(s_k, \boldsymbol{x}(s_k))| \geq \beta(\xi), \quad k \in \mathbb{N}. \tag{4.13}$$

Let us only consider the case that $\dot{V}_1(t, \boldsymbol{x})$ is bounded from above, the case when $\dot{V}_1(t, \boldsymbol{x})$ is bounded from below can be studied in a similar way. Then there is such a $K > 0$ that

$$\sup_{(t, \boldsymbol{x}) \in \mathfrak{G}} \dot{V}_1(t, \boldsymbol{x}) < K. \tag{4.14}$$

Let us choose such a $\varpi > 0$ that

$$\varpi < \min\left(\eta, \frac{\beta(\xi)}{2K}\right). \tag{4.15}$$

It follows from (4.13), (4.14) and (4.15) that for $t \in [s_k - \varpi, s_k]$

$$\begin{aligned}
V_1(t, \boldsymbol{x}(t)) &= V_1(s_k, \boldsymbol{x}(s_k)) + \int_{s_k}^{t} \dot{V}_1(s, \boldsymbol{x}(s))ds \\
&= V_1(s_k, \boldsymbol{x}(s_k)) - \int_{t}^{s_k} \dot{V}_1(s, \boldsymbol{x}(s))ds \\
&\geq \beta(\xi) - (s_k - t)K \qquad \Leftarrow (4.13)\ \&\ (4.14) \\
&\geq \beta(\xi) - \underbrace{\varpi K > \beta(\xi)/2}_{(4.15)}
\end{aligned} \tag{4.16}$$

from which and (4.9) we have

$$\begin{aligned}
0 &\leq V(s_k, \boldsymbol{x}(s_k)) \qquad \Leftarrow (4.7) \\
&= V(t_0^+, \boldsymbol{x}_0) + \int_{t_0}^{s_k} \dot{V}(s, \boldsymbol{x}(s))ds \\
&\leq V(t_0^+, \boldsymbol{x}_0) - \int_{t_0}^{s_k} \varrho(V_1(s, \boldsymbol{x}(s)))ds \qquad \Leftarrow (4.9) \\
&\leq V(t_0^+, \boldsymbol{x}_0) - \sum_{i=1}^{k} \int_{s_i - \varpi}^{s_i} \varrho(V_1(s, \boldsymbol{x}(s)))ds \qquad \Leftarrow (s_i - \varpi > s_{i-1}) \\
&\leq V(t_0^+, \boldsymbol{x}_0) - k\varpi\varrho(\beta(\xi)/2) \qquad \Leftarrow (4.16)
\end{aligned} \tag{4.17}$$

which leads to a contradiction when k is big enough. Thus, Claim 4.1.2 is true. It follows from Theorem 4.1.1 that the trivial solution of system (4.1) is stable. Then from Claim 4.1.2 we get conclusion 1.

Since $\Theta_a(t_0, \alpha)$ is a neighborhood of $\boldsymbol{x} = 0$, it follows from conclusion 1 that $\boldsymbol{x} = 0$ is attractive. Therefore, the trivial solution of system (4.1) is asymptotically stable. ■

From Theorem 4.1.2 we have the following corollary.

Corollary 4.1.1. *Assume that $V \in \mathcal{V}_1$ and there are $\alpha, \varrho \in \mathcal{K}$ such that*

$$\alpha(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}), \quad \text{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \tag{4.18}$$

$$\dot{V}(t, \boldsymbol{x}) \leq -\varrho(V(t, \boldsymbol{x})), \quad \text{for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{4.19}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x}), \ \text{for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho). \tag{4.20}$$

Then we have the following conclusions:

1. *If $0 < a < \rho$ and $t_0 \in \mathbb{R}_+$, then $\Xi(t_0) \supset \Theta_a(t_0, \alpha)$;*
2. *The trivial solution of system (4.1) is asymptotically stable.*

⊠

Proof. Let $V_1 = V$ and $\beta = \alpha$ then the proof is immediately followed from Theorem 4.1.2. ■

Theorem 4.1.3. *Assume that $V \in \mathcal{V}_1$ and there are $\alpha, \beta, \varrho \in \mathcal{K}$ such that*

$$\alpha(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|), \quad \textit{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \tag{4.21}$$

$$\dot{V}(t, \boldsymbol{x}) \leq -\varrho(\|\boldsymbol{x}\|), \quad \textit{for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{4.22}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x}) \leq V(t, \boldsymbol{x}), \ \textit{for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho). \tag{4.23}$$

Then we have the following conclusions:

1. *If $0 < a < \rho$, then*
$$\lim_{t \to \infty} \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| = 0$$
uniformly with respect to $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \Theta_a(t_0, \alpha)$;
2. *The trivial solution of system (4.1) is uniformly asymptotically stable.*

⊠

Proof. It follows from Corollary 4.1.1 that the two conclusions are satisfied without uniform properties. Then by using the same method we used in the proof of the second conclusion of Theorem 4.1.1 we can prove the uniform properties of both conclusions. ■

Theorem 4.1.4. *Assume that $V \in \mathcal{V}_1$ and there are positive constants a, b, c and $p \in \mathbb{N}$ such that*

$$a\|\boldsymbol{x}\|^p \leq V(t, \boldsymbol{x}) \leq b\|\boldsymbol{x}\|^p, \quad \textit{for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \tag{4.24}$$

$$\dot{V}(t, \boldsymbol{x}) \leq -c\|\boldsymbol{x}\|^p, \quad \textit{for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{4.25}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x}), \ \textit{for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \mathcal{S}_\rho). \tag{4.26}$$

Then the trivial solution of system (4.1) is exponentially stable. ⊠

Proof. Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (4.1) with $\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ such that $t_0 \in \mathbb{R}_+$ and $\boldsymbol{x}_0 \in \mathcal{S}_{\rho_0}$, then it follows from conditions in (4.24), (4.25) and (4.26) that

$$\begin{aligned} &\dot{V}(t, \boldsymbol{x}(t)) \leq -\frac{c}{b} V(t, \boldsymbol{x}(t)), \quad t \neq \tau_k, \\ &V(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \leq V(\tau_k, \boldsymbol{x}(\tau_k)), \quad k \in \mathbb{N}. \end{aligned} \tag{4.27}$$

Therefore we have

$$\begin{aligned} &V(t, \boldsymbol{x}(t)) \leq V(t_0^+, \boldsymbol{x}_0) \exp\left(-\frac{c}{b}(t - t_0)\right), \\ &\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| \leq \|\boldsymbol{x}_0\| \sqrt[p]{b/a} \exp\left(-\frac{c}{bp}(t - t_0)\right). \end{aligned} \tag{4.28}$$

Let us choose $\delta > 0$ such that $\delta \sqrt[p]{b/a} < \rho$, then (4.28) holds for any $\|\boldsymbol{x}\| < \delta$ and $t > t_0$. We then finish the proof. ■

Let us consider the following impulsive control system:

$$\boxed{\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}),\\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), t = \tau_k(\boldsymbol{x}), k \in \mathbb{N}\end{aligned} \qquad (4.29)}$$

where $t \in \mathbb{R}_+$ and $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$. We assume that there is no beating phenomenon on switching surfaces $t = \tau_k(\boldsymbol{x})$. We assume that $\boldsymbol{f}(t, 0) = 0$, $U(k, 0) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. For a $\rho_0 > 0$ let us define

$$\Omega \triangleq \{\boldsymbol{x} \mid \|\boldsymbol{x}\| \leq \rho_0\}$$

and assume $\boldsymbol{f} \in C[[0, \infty) \times \Omega, \mathbb{R}^n]$ and $U \in C[\mathbb{N} \times \Omega, \mathbb{R}^n]$.

Theorem 4.1.5. *Assume that a positive definite function $V(t, \boldsymbol{x})$ defined on $[0, \infty) \times \mathcal{S}_{\rho_0}$ satisfies*

$$\dot{V}(t, \boldsymbol{x}) \leq 0, \quad t \geq 0, \quad \boldsymbol{x} \in \Omega, \tag{4.30}$$

then the trivial solution of the control system (4.29) is

1. *stable if*

$$V(\tau_k(\boldsymbol{x}), \boldsymbol{x} + U(k, \boldsymbol{x})) - V(\tau_k(\boldsymbol{x}), \boldsymbol{x}) \leq 0, \quad k \in \mathbb{N}; \tag{4.31}$$

2. *asymptotically stable if*

$$V(\tau_k(\boldsymbol{x}), \boldsymbol{x} + U(k, \boldsymbol{x})) - V(\tau_k(\boldsymbol{x}), \boldsymbol{x}) \leq -\alpha(V(\tau_k(\boldsymbol{x}), \boldsymbol{x})) \tag{4.32}$$

where $k \in \mathbb{N}$, $\alpha \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\alpha(0) = 0$ and $\alpha(w) > 0$ for $w > 0$.

⊠

Proof.

1. Stability. Let us choose $\delta \in (0, \rho)$ and let

$$\xi = \inf_{t \geq 0, \|\boldsymbol{x}\| \in [\delta, \rho)} V(t, \boldsymbol{x})$$

and choose $\varepsilon > 0$ such that

$$\sup_{\|\boldsymbol{x}\| < \varepsilon} V(0, \boldsymbol{x}) = \eta < \xi.$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ be any a solution of system (4.29) such that $\boldsymbol{x}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0 \in \mathcal{S}_\varepsilon$. Let us assume that at $t = t_1$ we have $\|\boldsymbol{x}(t_1)\| = \delta$, then

$$V(t_1, \boldsymbol{x}(t_1)) \geq \xi. \tag{4.33}$$

It follows from the assumptions in (4.30) and (4.31) that $V(t, \boldsymbol{x}(t))$ is nonincreasing along any solution of impulsive control system (4.29) that lies in Ω. Therefore we have

$$V(t_1, \boldsymbol{x}(t_1)) \leq V(0, \boldsymbol{x}_0) \leq \eta < \xi \tag{4.34}$$

which is a contradiction to (4.33). This proves that for any $\delta > 0$ there is a $\varepsilon > 0$ such that if $\|\boldsymbol{x}_0\| \leq \varepsilon$ then $\|\boldsymbol{x}(t)\| < \delta$ for all $t \in \mathbb{R}_+$. This finishes the proof of the first conclusion.

2. Asymptotic stability. If (4.32) holds, then it follows from the first conclusion that the trivial solution is stable. To show that the trivial solution is asymptotically stable we only need to prove

$$\lim_{t \to \infty} V(t, \boldsymbol{x}(t)) = 0. \tag{4.35}$$

It follows from (4.30) and (4.32) that $V(t, \boldsymbol{x}(t))$ is nonincreasing. Then from the fact that $V(t, \boldsymbol{x}(t))$ is bounded from below we know that the following limit exists:

$$\lim_{t \to \infty} V(t, \boldsymbol{x}(t)) = \sigma.$$

We then prove that $\sigma > 0$ is impossible. Let us suppose that $\sigma > 0$ and let κ be

$$\kappa = \inf_{w \in [\sigma, V(0, \boldsymbol{x}_0)]} \alpha(w). \tag{4.36}$$

Suppose that the solution $\boldsymbol{x}(t)$ intersects the switching surface $t = \tau_k(\boldsymbol{x})$ at point $(t, \boldsymbol{x}) = (\tau_k, \boldsymbol{x}_k)$, it follows from (4.32) that

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) - V(\tau_k, \boldsymbol{x}_k) \leq -\alpha(V(\tau_k, \boldsymbol{x}_k)), \quad k \in \mathbb{N}. \tag{4.37}$$

Since $\sigma \leq V(\tau_k, \boldsymbol{x}_k) \leq V(0, \boldsymbol{x}_0)$, it follows from (4.36) that

$$-\alpha(V(\tau_k, \boldsymbol{x}_k)) \leq -\kappa$$

from which and (4.37) we obtain

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) - V(\tau_k, \boldsymbol{x}_k) \leq -\kappa. \tag{4.38}$$

It follows from (4.30) that $V(t, \boldsymbol{x}(t))$ is nonincreasing in $(\tau_k, \tau_{k+1}]$, we then have

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \geq V(\tau_{k+1}, \boldsymbol{x}_{k+1}). \tag{4.39}$$

Therefore, for any $K \in \mathbb{N}$ we have

$$
\begin{aligned}
&V(\tau_K^+, \boldsymbol{x}_K + U(K, \boldsymbol{x}_K)) \\
&\leq V(\tau_K^+, \boldsymbol{x}_K + U(K, \boldsymbol{x}_K)) + \underbrace{\sum_{k=0}^{K-1} [V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) - V(\tau_{k+1}, \boldsymbol{x}_{k+1})]}_{\geq 0 \text{ because of (4.39)}} \\
&= V(0, \boldsymbol{x}_0) + \sum_{k=1}^{K} [V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) - V(\tau_k, \boldsymbol{x}_k)] \\
&\leq V(0, \boldsymbol{x}_0) - K\kappa \qquad \Leftarrow (4.38)
\end{aligned}
\tag{4.40}
$$

which leads to $V(\tau_K^+, \boldsymbol{x}_K + U(K, \boldsymbol{x}_K)) < 0$ for large enough K. This is a contradiction to the assumption that $V(t, \boldsymbol{x})$ is positive definite. Therefore, $\sigma > 0$ is not true and (4.35) is valid.

■

Theorem 4.1.6. *Assume that a positive definite function $V(t, \boldsymbol{x})$ defined on $\mathbb{R}_+ \times \mathcal{S}_{\rho_0}$ satisfies*

1.

$$\dot{V}(t, \boldsymbol{x}) \leq -\beta(V(t, \boldsymbol{x})), \tag{4.41}$$

$$
\begin{aligned}
&V(\tau_k^+(\boldsymbol{x}), \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \alpha(V(\tau_k(\boldsymbol{x}), \boldsymbol{x})), \\
&t \geq 0, \quad \boldsymbol{x} \in \Omega, \quad k \in \mathbb{N}
\end{aligned}
\tag{4.42}
$$

where $\alpha, \beta \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\alpha(0) = \beta(0) = 0$ and $\alpha(w) > 0, \beta(w) > 0$ for $w > 0$;

2.

$$\sup_{k \in \mathbb{N}} \left(\min_{\|\boldsymbol{x}\| \leq \rho} \tau_{k+1}(\boldsymbol{x}) - \max_{\|\boldsymbol{x}\| \leq \rho} \tau_k(\boldsymbol{x}) \right) = \phi > 0. \tag{4.43}$$

Then the trivial solution of the control system (4.29) is

1. stable if

$$\int_{\epsilon}^{\alpha(\epsilon)} \frac{1}{\beta(s)} ds \leq \phi \tag{4.44}$$

for some $\epsilon_0 > 0$ and all $\epsilon \in (0, \epsilon_0]$;

2. asymptotically stable if

$$\int_{\epsilon}^{\alpha(\epsilon)} \frac{1}{\beta(s)} ds \leq \phi - \zeta \tag{4.45}$$

for some $\zeta > 0$.

⊠

Proof.

1. Stability. Let us choose $\delta > 0$ and let

$$\xi = \inf_{t\geq 0, \|\boldsymbol{x}\|\geq\delta} V(t,\boldsymbol{x})$$

and choose $\varepsilon > 0$ such that

$$\sup_{\|\boldsymbol{x}\|<\varepsilon} V(0,\boldsymbol{x}) = \eta < \xi.$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t,\boldsymbol{x}_0)$ be any a solution of system (4.29) such that $\boldsymbol{x}(0,\boldsymbol{x}_0) = \boldsymbol{x}_0 \in \mathcal{S}_\varepsilon$. To prove that the trivial solution is stable we need to prove that $\boldsymbol{x}(t)$ will stay within $\mathcal{S}_\delta$ for all $t \in \mathbb{R}_+$. To prove this we can prove that $V(t,\boldsymbol{x}) < \xi$ for all $t \in \mathbb{R}_+$.
Let us assume that $\boldsymbol{x}(t)$ intersects the switching surface $t = \tau_1(\boldsymbol{x})$ at point $(\tau_1, \boldsymbol{x}_1)$. It follows from (4.41) that

$$\dot{V}(t,\boldsymbol{x}(t)) \leq -\beta(V(t,\boldsymbol{x}(t))) \text{ for } t \in [0,\tau_1].$$

Therefore we have

$$-\int_0^{\tau_1} \frac{\dot{V}(t,\boldsymbol{x}(t))}{\beta(V(t,\boldsymbol{x}(t)))} dt \geq \tau_1.$$

Let us substitute a new variable $s \triangleq V(t,\boldsymbol{x}(t))$ into this inequality and from (4.43) we have

$$\begin{aligned}
-\int_0^{\tau_1} \frac{\dot{V}(t,\boldsymbol{x}(t))}{\beta(V(t,\boldsymbol{x}(t)))} dt &= \int_{\tau_1}^{0} \frac{\dot{V}(t,\boldsymbol{x}(t))}{\beta(V(t,\boldsymbol{x}(t)))} dt \\
&= \int_{V(\tau_1,\boldsymbol{x}_1)}^{V(0,\boldsymbol{x}_0)} \frac{1}{\beta(s)} ds \qquad \Leftarrow s = V(t,\boldsymbol{x}(t)) \\
&\geq \tau_1 - 0 \geq \phi. \qquad \Leftarrow (4.43)
\end{aligned} \tag{4.46}$$

It follows from (4.42) and (4.44) that

$$\int_{V(\tau_1,\boldsymbol{x}_1)}^{V(\tau_1^+,\boldsymbol{x}_1+U(1,\boldsymbol{x}_1))} \underbrace{\frac{1}{\beta(s)} ds \leq \int_{V(\tau_1,\boldsymbol{x}_1)}^{\alpha(V(\tau_1,\boldsymbol{x}_1))}}_{(4.42)} \underbrace{\frac{1}{\beta(s)} ds \leq \phi}_{(4.44)}. \tag{4.47}$$

It follows from (4.46) and (4.47) that

$$\begin{aligned}
&\int_{V(\tau_1^+,\boldsymbol{x}_1+U(1,\boldsymbol{x}_1))}^{V(0,\boldsymbol{x}_0)} \frac{1}{\beta(s)} ds \\
&= \int_{V(\tau_1,\boldsymbol{x}_1)}^{V(0,\boldsymbol{x}_0)} \frac{1}{\beta(s)} ds - \int_{V(\tau_1,\boldsymbol{x}_1)}^{V(\tau_1^+,\boldsymbol{x}_1+U(1,\boldsymbol{x}_1))} \frac{1}{\beta(s)} ds \geq 0
\end{aligned} \tag{4.48}$$

from which we have

$$V(\tau_1^+, \boldsymbol{x}_1 + U(1, \boldsymbol{x}_1)) \leq V(0, \boldsymbol{x}_0).$$

Then by using mathematical induction we know that

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \leq V(0, \boldsymbol{x}_0)$$

for all $k \in \mathbb{N}$. Therefore we have the first conclusion.

2. Asymptotic stability. If (4.45) holds, then from the first conclusion we know that the trivial solution is stable. Let us assume that $\boldsymbol{x}(t)$ intersects the switching surface $t = \tau_k(\boldsymbol{x})$ at $(\tau_k, \boldsymbol{x}_k)$. Then it follows from (4.41) that

$$-\int_{\tau_k^+}^{\tau_{k+1}} \frac{\dot{V}(t, \boldsymbol{x})}{\beta(V(t, \boldsymbol{x}))} dt \geq \tau_{k+1} - \tau_k \geq \phi. \tag{4.49}$$

Let us substitute ϵ in (4.45) with $V(\tau_{k+1}, \boldsymbol{x}_{k+1})$ and in view of (4.42) we have

$$\begin{aligned} &\int_{V(\tau_{k+1}, \boldsymbol{x}_{k+1})}^{V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1}))} \underbrace{\frac{1}{\beta(s)} \leq \int_{V(\tau_{k+1}, \boldsymbol{x}_{k+1})}^{\alpha(V(\tau_{k+1}, \boldsymbol{x}_{k+1})}}_{(4.42)} \frac{1}{\beta(s)} \\ &\leq \phi - \zeta. \qquad \Leftarrow (4.45) \end{aligned} \tag{4.50}$$

It follows from (4.49) and (4.50) that

$$\begin{aligned} \int_{V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1}))}^{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))} \frac{1}{\beta(s)} &= \int_{V(\tau_{k+1}, \boldsymbol{x}_{k+1})}^{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))} \frac{1}{\beta(s)} \\ &\quad - \int_{V(\tau_{k+1}, \boldsymbol{x}_{k+1})}^{V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1}))} \frac{1}{\beta(s)} \\ &\geq \phi - (\phi - \zeta) = \zeta. \end{aligned} \tag{4.51}$$

Therefore for the sequence $\{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))\}$ we have

$$\int_{V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1}))}^{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))} \frac{1}{\beta(s)} \geq \zeta, \quad k \in \mathbb{N} \tag{4.52}$$

from which we know that the sequence $\{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))\}$ is decreasing as $k \to \infty$. We then have the following claim:
Claim 4.1.6:

$$\lim_{k \to \infty} V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) = 0.$$

If Claim 4.1.6 is not true then we can assume that

$$\lim_{k\to\infty} V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) = \sigma > 0.$$

Let

$$\kappa = \inf_{s\in[\sigma, V(t,\boldsymbol{x}_0)]} \beta(s)$$

then it follows from (4.52) that

$$\begin{aligned}\zeta &\le \int_{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}^{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))} \frac{1}{\beta(s)} \\ &\le \frac{1}{\kappa}[V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \\ &\quad -V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1}))]\end{aligned} \tag{4.53}$$

from which we know that

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) - V(\tau_{k+1}^+, \boldsymbol{x}_{k+1} + U(k+1, \boldsymbol{x}_{k+1})) \ge \zeta\kappa. \tag{4.54}$$

This is a contradiction to the convergence of sequence $\{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))\}$. Therefore, Claim 4.1.6 is true.
It follows from (4.41) that $V(t, \boldsymbol{x}(t))$ is decreasing in every $(\tau_k, \tau_{k+1}]$ for $k \in \mathbb{N}$. Thus we have

$$\sup_{t\in(\tau_k,\tau_{k+1})} V(t, \boldsymbol{x}(t)) = V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \tag{4.55}$$

from which and $V(\tau_{k+1}^+, \boldsymbol{x}_{k+1}+U(k+1, \boldsymbol{x}_{k+1})) < V(\tau_k^+, \boldsymbol{x}_k+U(k, \boldsymbol{x}_k)), k \in \mathbb{N}$, we know that

$$V(t, \boldsymbol{x}(t)) < V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \text{ for all } t > \tau_k. \tag{4.56}$$

Then it follows from Claim 4.1.6 that

$$\lim_{t\to\infty} V(t, \boldsymbol{x}(t)) = 0 \tag{4.57}$$

from which we know that

$$\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0. \tag{4.58}$$

This finishes the proof of the second conclusion.

■

Similarly, we have the following theorem.

Theorem 4.1.7. *Assume that a positive definite function $V(t, \boldsymbol{x})$ defined on $\mathbb{R}_+ \times \mathcal{S}_{\rho_0}$ satisfies*

1.

$$\dot{V}(t,\boldsymbol{x}) \le \beta(V(t,\boldsymbol{x})), \tag{4.59}$$

$$V(\tau_k^+(\boldsymbol{x}), \boldsymbol{x}+U(k,\boldsymbol{x})) \le \alpha(V(\tau_k(\boldsymbol{x}),\boldsymbol{x})),$$
$$t \ge 0, \quad \boldsymbol{x} \in \mathcal{S}_{\rho_0}, \quad k \in \mathbb{N} \tag{4.60}$$

where $\alpha, \beta \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\alpha(0) = \beta(0) = 0$ *and* $\alpha(w) > 0, \beta(w) > 0$ *for* $w > 0$*;*

2.

$$\max_{\|\boldsymbol{x}\|\le\rho} \tau_{k+1}(\boldsymbol{x}) - \min_{\|\boldsymbol{x}\|\le\rho} \tau_k(\boldsymbol{x}) \le \phi, \quad \phi > 0, \quad k \in \mathbb{N}. \tag{4.61}$$

Then the trivial solution of the control system (4.29) is

1. *stable if*

$$\int_{\alpha(\epsilon)}^{\epsilon} \frac{1}{\beta(s)} ds \ge \phi \tag{4.62}$$

for some $\epsilon_0 > 0$ *and all* $\epsilon \in (0, \epsilon_0]$*;*

2. *asymptotically stable if*

$$\int_{\alpha(\epsilon)}^{\epsilon} \frac{1}{\beta(s)} ds \ge \phi + \zeta \tag{4.63}$$

for some $\zeta > 0$.

$\boxtimes$

Proof.

1. Stability. Let us choose $\delta > 0$ and let

$$\xi = \inf_{t\ge 0, \|\boldsymbol{x}\|\ge\delta} V(t,\boldsymbol{x})$$

and choose $\varepsilon > 0$ such that

$$\sup_{\|\boldsymbol{x}\|<\varepsilon} V(0,\boldsymbol{x}) = \eta < \xi.$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t,\boldsymbol{x}_0)$ be any a solution of system (4.29) such that $\boldsymbol{x}(0,\boldsymbol{x}_0) = \boldsymbol{x}_0 \in \mathcal{S}_\varepsilon$. To prove that the trivial solution is stable we need to prove that $\boldsymbol{x}(t)$ will stay within $\mathcal{S}_\delta$ for all $t \in \mathbb{R}_+$. To prove this we can prove that $V(t,\boldsymbol{x}) < \xi$ for all $t \in \mathbb{R}_+$.
Let us assume that $\boldsymbol{x}(t)$ intersects the switching surface $t = \tau_1(\boldsymbol{x})$ at point $(\tau_1, \boldsymbol{x}_1)$. It follows from (4.59) that

$$\dot{V}(t, \boldsymbol{x}(t)) \leq \beta(V(t, \boldsymbol{x}(t))) \text{ for } t \in [0, \tau_1].$$

Therefore we have

$$\int_0^{\tau_1} \frac{\dot{V}(t, \boldsymbol{x}(t))}{\beta(V(t, \boldsymbol{x}(t)))} dt \leq \tau_1.$$

Let us substitute a new variable $s \triangleq V(t, \boldsymbol{x}(t))$ into this inequality and from (4.61) we have

$$\begin{aligned}
\int_0^{\tau_1} \frac{\dot{V}(t, \boldsymbol{x}(t))}{\beta(V(t, \boldsymbol{x}(t)))} dt &= -\int_{\tau_1}^{0} \frac{\dot{V}(t, \boldsymbol{x}(t))}{\beta(V(t, \boldsymbol{x}(t)))} dt \\
&= -\int_{V(\tau_1, \boldsymbol{x}_1)}^{V(0, \boldsymbol{x}_0)} \frac{1}{\beta(s)} ds \qquad \Leftarrow [s = V(t, \boldsymbol{x}(t))] \\
&\leq \tau_1 - 0 \leq \phi. \qquad \Leftarrow (4.61)
\end{aligned} \tag{4.64}$$

It follows from (4.60) and (4.62) that

$$\int_{V(\tau_1, \boldsymbol{x}_1)}^{V(\tau_1^+, \boldsymbol{x}_1 + U(1, \boldsymbol{x}_1))} \underbrace{\frac{1}{\beta(s)} ds \leq \int_{V(\tau_1, \boldsymbol{x}_1)}^{\alpha(V(\tau_1, \boldsymbol{x}_1))}}_{(4.60)} \underbrace{\frac{1}{\beta(s)} ds \leq -\phi}_{(4.62)}. \tag{4.65}$$

It follows from (4.64) and (4.65) that

$$\begin{aligned}
&\int_{V(\tau_1^+, \boldsymbol{x}_1 + U(1, \boldsymbol{x}_1))}^{V(0, \boldsymbol{x}_0)} \frac{1}{\beta(s)} ds \\
&= \int_{V(\tau_1, \boldsymbol{x}_1)}^{V(0, \boldsymbol{x}_0)} \frac{1}{\beta(s)} ds - \int_{V(\tau_1, \boldsymbol{x}_1)}^{V(\tau_1^+, \boldsymbol{x}_1 + U(1, \boldsymbol{x}_1))} \frac{1}{\beta(s)} ds \geq 0
\end{aligned} \tag{4.66}$$

from which we have

$$V(\tau_1^+, \boldsymbol{x}_1 + U(1, \boldsymbol{x}_1)) \leq V(0, \boldsymbol{x}_0).$$

Then by using mathematical induction we know that

$$V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k)) \leq V(0, \boldsymbol{x}_0)$$

for all $k \in \mathbb{N}$. Therefore we have the first conclusion.

2. Asymptotic stability. If (4.63) holds, then from the first conclusion of this theorem we know that the trivial solution is stable. Let us assume that $\boldsymbol{x}(t)$ intersects the switching surface $t = \tau_k(\boldsymbol{x})$ at $(\tau_k, \boldsymbol{x}_k)$. Then it follows from (4.59) that

$$\int_{\tau_k^+}^{\tau_{k+1}} \frac{\dot{V}(t, \boldsymbol{x})}{\beta(V(t, \boldsymbol{x}))} dt \leq \tau_{k+1} - \tau_k \leq \phi. \tag{4.67}$$

Substituting ϵ in (4.63) with $V(\tau_{k+1},\boldsymbol{x}_{k+1})$ and in view of (4.60) we have

$$\int_{V(\tau_{k+1},\boldsymbol{x}_{k+1})}^{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}\underbrace{\frac{1}{\beta(s)}\le\int_{V(\tau_{k+1},\boldsymbol{x}_{k+1})}^{\alpha(V(\tau_{k+1},\boldsymbol{x}_{k+1})}}_{(4.60)}\frac{1}{\beta(s)}$$
$$\le -\phi-\zeta. \qquad \Leftarrow (4.63) \tag{4.68}$$

It follows from (4.67) and (4.68) that

$$\begin{aligned}\int_{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}^{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))}\frac{1}{\beta(s)} &= \int_{V(\tau_{k+1},\boldsymbol{x}_{k+1})}^{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))}\frac{1}{\beta(s)}\\ &\quad -\int_{V(\tau_{k+1},\boldsymbol{x}_{k+1})}^{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}\frac{1}{\beta(s)}\\ &\ge -\phi-(-\phi-\zeta)=\zeta.\end{aligned} \tag{4.69}$$

Therefore for the sequence $\{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))\}$ we have

$$\int_{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}^{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))}\frac{1}{\beta(s)}\ge\zeta,\quad k\in\mathbb{N} \tag{4.70}$$

from which we know that the sequence $\{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))\}$ is decreasing as $k\to\infty$. We then have the following claim:
Claim 4.1.7:

$$\lim_{k\to\infty}V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))=0.$$

If Claim 4.1.7 is not true then we can assume that

$$\lim_{k\to\infty}V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))=\sigma>0.$$

Let

$$\kappa=\inf_{s\in[\sigma,V(t,\boldsymbol{x}_0)]}\beta(s)$$

then it follows from (4.70) that

$$\begin{aligned}\zeta &\le \int_{V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))}^{V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))}\frac{1}{\beta(s)}\\ &\le \frac{1}{\kappa}[V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))\\ &\quad -V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))]\end{aligned} \tag{4.71}$$

from which we know that

$$V(\tau_k^+,\boldsymbol{x}_k+U(k,\boldsymbol{x}_k))-V(\tau_{k+1}^+,\boldsymbol{x}_{k+1}+U(k+1,\boldsymbol{x}_{k+1}))\ge\zeta\kappa. \tag{4.72}$$

This is a contradiction to the convergence of sequence $\{V(\tau_k^+, \boldsymbol{x}_k + U(k, \boldsymbol{x}_k))\}$. Therefore, Claim 4.1.7 is true.
Similarly, we can prove that

$$\lim_{t\to\infty} V(t, \boldsymbol{x}(t)) = 0 \tag{4.73}$$

from which we know that

$$\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0. \tag{4.74}$$

This finishes the proof of the second conclusion.

■

Example 4.1.1. Let us study the stability of the trivial solution of the following impulsive control system:

$$\left.\begin{array}{l} \dot{x} = y, \\ \dot{y} = -\sin x \end{array}\right\}, \quad t \neq \tau_k(x, y),$$
$$\left.\begin{array}{l} \Delta x = -x + \cos^{-1}\left(\cos(x) - \frac{y^2}{2}\right) \\ \Delta y = -y \end{array}\right\}, \quad t = \tau_k(x, y). \tag{4.75}$$

Let us construct a Lyapunov function as

$$V(x, y) = 1 - \cos x + \frac{y^2}{2}. \tag{4.76}$$

Then we have

$$\dot{V}(x, y) = y \sin x - y \sin x = 0 \tag{4.77}$$

and

$$\begin{aligned} V(x + \Delta x, y + \Delta y) &= 1 - \cos\left(\cos^{-1}\left(\cos(x) - \frac{y^2}{2}\right)\right) \\ &= 1 - \cos(x) + \frac{y^2}{2} = V(x, y). \end{aligned} \tag{4.78}$$

It is clear that for any kind of switching surface $t = \tau_k(x, y)$, the conditions of Theorem 4.1.5 hold. Therefore, the trivial solution of system (4.75) is stable. ★

4.2 Linear Decomposition Methods

In this section we consider the stability of the zero solution of the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{g}(t,\boldsymbol{x}), & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x} + U(k,\boldsymbol{x}), \; t = \tau_k, & k \in \mathbb{N} \end{cases} \tag{4.79}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$, $\boldsymbol{g} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{g}(t,0) = 0$, $B_k \in \mathbb{R}^{n\times n}$, $U : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ and $U(k,0) = 0, k \in \mathbb{N}$. The plant is decomposed into linear and nonlinear parts. $\boldsymbol{g}(t,\boldsymbol{x})$ can be the nonlinearity of the plant or the nonlinearity of the continuous control law. We assume that

$$\tau_{k+1} - \tau_k \geq \theta, \qquad k \in \mathbb{N}, \quad \theta > 0. \tag{4.80}$$

We also need the following reference system

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x}, \; t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x}, \; t = \tau_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.81}$$

Theorem 4.2.1. *Assume that*

1. *the state transition matrix $\Psi(t,s)$ of reference system (4.81) satisfy*

$$\|\Psi(t,s)\| \leq Ke^{-\gamma(t-s)}, \quad K \geq 1, \quad \gamma > 0 \tag{4.82}$$

for all t and s with $t_0 \leq s \leq t$;

2. *for all $t \geq t_0$, $k \in \mathbb{N}$, $\|\boldsymbol{x}\| \leq h$, $h > 0$ we have*

$$\|\boldsymbol{g}(t,\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|U(k,\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|; \tag{4.83}$$

3. $$\gamma - Ka - \frac{1}{\theta}\ln(1+Ka) > 0$$

where θ is given by (4.80).

Then the zero solution of system (4.79) is asymptotically stable. ⊠

Proof. Let the initial condition be $\boldsymbol{x}(t_0,\boldsymbol{x}_0) = \boldsymbol{x}_0$ then every solution of system (4.79) can be written as

$$\begin{aligned} \boldsymbol{x}(t,\boldsymbol{x}_0) = \Psi(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Psi(t,s)\boldsymbol{g}(s,\boldsymbol{x}(s,\boldsymbol{x}_0))ds \\ + \sum_{t_0<\tau_i<t} \Psi(t,\tau_i)U(i,\boldsymbol{x}(\tau_i,\boldsymbol{x}_0)). \end{aligned} \tag{4.84}$$

From (4.82) and (4.83) we have

$$\begin{aligned} \|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \leq Ke^{-\gamma(t-t_0)}\|\boldsymbol{x}_0\| + \int_{t_0}^{t} Ke^{-\gamma(t-s)}a\|\boldsymbol{x}(s,\boldsymbol{x}_0)\|ds \\ + \sum_{t_0<\tau_i<t} Ke^{-\gamma(t-\tau_i)}a\|\boldsymbol{x}(\tau_i,\boldsymbol{x}_0)\|, \end{aligned} \tag{4.85}$$

or in the following form:

$$e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \leq K\|\boldsymbol{x}_0\| + \int_{t_0}^{t} Kae^{\gamma(s-t_0)}\|\boldsymbol{x}(s,\boldsymbol{x}_0)\|ds + \sum_{t_0<\tau_i<t} Kae^{\gamma(\tau_i-t_0)}\|\boldsymbol{x}(\tau_i,\boldsymbol{x}_0)\|. \tag{4.86}$$

By using Lemma 1.7.1, we have

$$e^{\gamma(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \leq K\|\boldsymbol{x}_0\|(1+Ka)^{\mathfrak{N}(t_0,t)}e^{Ka(t-t_0)}. \tag{4.87}$$

From the assumption (4.80) it follows

$$\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \leq K\|\boldsymbol{x}_0\| \exp\left\{-\left(\gamma - Ka - \frac{1}{\theta}\ln(1+Ka)\right)(t-t_0)\right\}. \tag{4.88}$$

Then from assumption 3 it follows that for all $\|\boldsymbol{x}_0\| < h/K$,

$$\lim_{t\to\infty}\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| = 0.$$

■

Theorem 4.2.2. *Assume that*

1. *the largest eigenvalue of matrix* $\frac{1}{2}(A(t)+A^\top(t))$, $\lambda_n(t)$, *satisfies* $\lambda_n(t) \leq \gamma$ *for all* $t \geq t_0$ *and the largest eigenvalue of the matrix* $(I+B_i^\top)(I+B_i)$, Λ_i, *satisfies* $\Lambda_i \leq \alpha^2$ *for all* $i \in \mathbb{N}$*;*
2. *the limit*
$$\lim_{T\to\infty}\frac{\mathfrak{N}(t,t+T)}{T} = p$$
exists and is uniform for all $t \geq t_0$*;*
3. $\gamma + p\ln\alpha < 0$*;*
4. *for all* $t \geq t_0$, $k \in \mathbb{N}$, $\|\boldsymbol{x}\| \leq h$, $h > 0$ *we have*

$$\|\boldsymbol{g}(t,\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|U(k,\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{4.89}$$

Then the zero solution of system (4.79) is asymptotically stable with a sufficiently small constant a. ⊠

Proof. Let the initial condition be $\boldsymbol{x}_r(t_0,\boldsymbol{x}_0) = \boldsymbol{x}_0$ then every solution of reference system (4.81) satisfies

$$\|\boldsymbol{x}_r(t_0,\boldsymbol{x}_0)\| \leq \alpha^{\mathfrak{N}(t_0,t)}e^{\gamma(t-t_0)}\|\boldsymbol{x}_0\|. \tag{4.90}$$

Based on assumptions 2 and 3 and (4.90) we can find $K \geq 1$ and $\mu > 0$ with $0 < \mu < |\gamma + p\ln\alpha|$ such that for all $t_0 \leq s \leq t$, the state transition matrix of system (4.81), $\Psi(t,s)$, satisfies

$$\|\Psi(t,s)\boldsymbol{z}\| \le Ke^{-\mu(t-s)}\|\boldsymbol{z}\|, \qquad \boldsymbol{z} \in \mathbb{R}^n. \tag{4.91}$$

In a sufficiently small neighborhood of the point $\boldsymbol{x}_0$ the solution of system (4.79) is given by

$$\begin{aligned}\boldsymbol{x}(t,\boldsymbol{x}_0) = \Psi(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Psi(t,s)\boldsymbol{g}(s,\boldsymbol{x}(s,\boldsymbol{x}_0))ds \\ + \sum_{t_0<\tau_i<t} \Psi(t,\tau_i)U(i,\boldsymbol{x}(\tau_i,\boldsymbol{x}_0)).\end{aligned} \tag{4.92}$$

Using the similar process in the proof of Theorem 4.2.1 we have

$$e^{\mu(t-t_0)}\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \le K\|\boldsymbol{x}_0\|(1+Ka)^{\mathfrak{N}(t_0,t)}e^{Ka(t-t_0)}, \tag{4.93}$$

from which we have

$$\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \le K_1\|\boldsymbol{x}_0\| \exp\{-(\mu - p\ln(1+Ka) - Ka + \epsilon)(t-t_0)\} \tag{4.94}$$

for any $\epsilon > 0$ with $K_1 = K_1(\epsilon) > 0$. Therefore, if a is sufficiently small such that

$$\mu - p\ln(1+Ka) - Ka > 0$$

then $\|\boldsymbol{x}(t,\boldsymbol{x}_0)\| \to 0$ as $t \to 0$. ■

Similarly, we have the following theorem.

Theorem 4.2.3. *Assume that*

1. *the largest eigenvalue of matrix* $\frac{1}{2}(A(t)+A^\top(t))$, $\lambda_n(t)$, *satisfies* $\lambda_n(t) \le \gamma$ *for all* $t \ge t_0$ *and the largest eigenvalue of the matrix* $(I+B_i^\top)(I+B_i)$, Λ_i, *satisfies* $\Lambda_i \le \alpha^2$ *for all* $i \in \mathbb{N}$*;*
2. $\tau_k, k \in \mathbb{N}$ *satisfy* $0 < \theta_1 \le \tau_k - \tau_{k-1} \le \theta_2$*;*
3. $$\gamma + \frac{1}{\theta}\ln\alpha < 0,$$
   where
   $$\theta = \begin{cases} \theta_2, & \text{if } 0<\alpha<1, \\ \theta_1, & \text{if } \alpha \ge 1; \end{cases}$$
4. *for all* $t \ge t_0$, $k \in \mathbb{N}$, $\|\boldsymbol{x}\| \le h$, $h > 0$ *we have*

$$\|\boldsymbol{g}(t,\boldsymbol{x})\| \le a\|\boldsymbol{x}\|, \qquad \|U(k,\boldsymbol{x})\| \le a\|\boldsymbol{x}\|. \tag{4.95}$$

Then the zero solution of system (4.79) is asymptotically stable with a sufficiently small constant a. ⊠

If in system (4.79) $A(t)$ and $B_k, k \in \mathbb{N}$ are time-invariant, then system (4.79) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x} + U(k, \boldsymbol{x}), & t = \tau_k, \quad k \in \mathbb{N} \end{cases} \tag{4.96}$$

where $\boldsymbol{g}(t, 0) = 0$ and $U(k, 0) = 0, k \in \mathbb{N}$. We then have the following theorems.

Theorem 4.2.4. *Assume that*

1. *A is in real canonical form and*

$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A), \quad \alpha^2 = \max_{j=1}^{n} \mathrm{Re}\lambda_j[(I + B^\top)(I + B)]$$

   where $\lambda_j(A)$ and $\lambda_j[(I + B^\top)(I + B)]$ are the j-th eigenvalues of A and $(I + B^\top)(I + B)$, respectively;
2. *the limit*

$$\lim_{T \to \infty} \frac{\mathfrak{N}(t, t + T)}{T} = p$$

   exists and is uniform for all $t \geq t_0$;
3. $\gamma + p \ln \alpha < 0$;
4. *for all $t \geq t_0$, $k \in \mathbb{N}$, $\|\boldsymbol{x}\| \leq h$, $h > 0$ we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \quad \|U(k, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{4.97}$$

Then the zero solution of system (4.96) is asymptotically stable with a sufficiently small constant a. ⊠

Theorem 4.2.5. *Assume that*

1. *A is in real canonical form and*

$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A), \quad \alpha^2 = \max_{j=1}^{n} \mathrm{Re}\lambda_j[(I + B^\top)(I + B)];$$

2. $\tau_k, k \in \mathbb{N}$ *satisfy* $0 < \theta_1 \leq \tau_k - \tau_{k-1} \leq \theta_2$;
3.

$$\gamma + \frac{1}{\theta} \ln \alpha < 0,$$

   where

$$\theta = \begin{cases} \theta_2, & \text{if } 0 < \alpha < 1, \\ \theta_1, & \text{if } \alpha \geq 1; \end{cases}$$

4. *for all $t \geq t_0$, $k \in \mathbb{N}$, $\|\boldsymbol{x}\| \leq h$, $h > 0$ we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|U(k, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{4.98}$$

Then the zero solution of system (4.96) is asymptotically stable with a sufficiently small constant a. ⊠

4.3 Methods Based on Linearization

In this section, we consider the stability of the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), & t \neq \tau_k, \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & t_0 \geq 0, \quad k = 1, 2, \cdots, \end{cases} \tag{4.99}$$

where $\boldsymbol{f}(0) = 0$, $U(k, 0) = 0$ and $\boldsymbol{f} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$. Let matrix $A \in \mathbb{R}^{n\times n}$ be the Jacobian of $\boldsymbol{f}$ at $\boldsymbol{x} = 0$; namely,

$$A = \begin{bmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} & \cdots & \frac{\partial f_1(0)}{\partial x_n} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} & \cdots & \frac{\partial f_2(0)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(0)}{\partial x_1} & \frac{\partial f_n(0)}{\partial x_2} & \cdots & \frac{\partial f_n(0)}{\partial x_n} \end{bmatrix} \tag{4.100}$$

then the linearized version of system (4.99) is given by

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x}, & t \neq \tau_k, \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & t_0 \geq 0, \quad k = 1, 2, \cdots, \end{cases} \tag{4.101}$$

Theorem 4.3.1. *Assume that*

1. *A is negative definite and there is a symmetric positive definite matrix P that is the unique solution of the following equation:*

$$A^\top P + PA = -Q. \tag{4.102}$$

Let λ_1 and λ_2 be the smallest and the largest eigenvalues of P and λ_3 be the smallest eigenvalue of $P^{-1}Q$ such that

$$-\lambda_3 + 2\sqrt{\frac{\lambda_2}{\lambda_1}} < 0; \tag{4.103}$$

2.

$$[\boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x})]^\top P[\boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x})] \leq \boldsymbol{x}(\tau_k)^\top P\boldsymbol{x}(\tau_k).$$

Then the origin $\boldsymbol{x} = 0$ of system (4.99) is asymptotically stable. ⊠

Proof. Since $\boldsymbol{f} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ and $\boldsymbol{f}(0) = 0$ we have

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}) \tag{4.104}$$

where $\boldsymbol{g} \in C[\mathbb{R}^n, \mathbb{R}^n]$ satisfies

$$\lim_{\|\boldsymbol{x}\| \to 0} \frac{\|\boldsymbol{g}(\boldsymbol{x})\|}{\|\boldsymbol{x}\|} = 0 \tag{4.105}$$

from which we know that there is a $\delta > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{x})\| \leq \|\boldsymbol{x}\| \text{ provided } \|\boldsymbol{x}\| \leq \delta. \tag{4.106}$$

For any $t \in (\tau_k, \tau_{k+1}]$ we have

$$\boldsymbol{x}(t) = \boldsymbol{x}(\tau_k^+) + \int_{\tau_k}^{t} [A\boldsymbol{x}(s) + \boldsymbol{g}(\boldsymbol{x}(s))]ds \tag{4.107}$$

from which we have

$$\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{x}(\tau_k^+)\| + \int_{\tau_k}^{t} [\|A\|\|\boldsymbol{x}(s)\| + \|\boldsymbol{g}(\boldsymbol{x}(s))\|]ds, \ t \in (\tau_k, \tau_{k+1}] \tag{4.108}$$

We then have the following claim:
Claim 4.3.1: Let $\delta_1 = \delta e^{-(\|A\|+1)(\tau_{k+1}-\tau_k)}$, then $\|\boldsymbol{x}(t)\| \leq \delta$ for all $t \in (\tau_k, \tau_{k+1}]$ whenever $\boldsymbol{x}(\tau_k^+) \leq \delta_1$.

If Claim 4.3.1 is not true then there is a $t^* \in (\tau_k, \tau_{k+1}]$ such that $\|\boldsymbol{x}(t^*)\| = \delta$ and $\|\boldsymbol{x}(t)\| \leq \delta$ for all $t \in (\tau_k, t^*]$.

Since $\|\boldsymbol{x}(t)\| \leq \delta$ for all $t \in (\tau_k, t^*]$ by assumption and in view of (4.106) and (4.108) we have

$$\begin{aligned}
\|\boldsymbol{x}(t)\| &\leq \|\boldsymbol{x}(\tau_k^+)\| + \int_{\tau_k}^{t} [\|A\|\|\boldsymbol{x}(s)\| + \|\boldsymbol{g}(\boldsymbol{x}(s))\|]ds \qquad \Leftarrow (4.108) \\
&\leq \|\boldsymbol{x}(\tau_k^+)\| + \int_{\tau_k}^{t} [\|A\|\|\boldsymbol{x}(s)\| + \|\boldsymbol{x}(s)\|]ds, \qquad \Leftarrow (4.106) \\
&\quad t \in (\tau_k, t^*].
\end{aligned} \tag{4.109}$$

It follows from Gronwell-Bellman inequalities that (4.109) implies

$$\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{x}(\tau_k^+)\| e^{(\|A\|+1)(t-\tau_k)}, \qquad t \in (\tau_k, t^*]. \tag{4.110}$$

Therefore

$$\begin{aligned}
\|\boldsymbol{x}(t^*)\| &\leq \|\boldsymbol{x}(\tau_k^+)\| e^{(\|A\|+1)(t^*-\tau_k)} \\
&\leq \delta_1 e^{(\|A\|+1)(t^*-\tau_k)} \\
&= \delta e^{-(\|A\|+1)(\tau_{k+1}-\tau_k)} e^{(\|A\|+1)(t^*-\tau_k)} \\
&= \delta e^{(\|A\|+1)(t^*-\tau_{k+1})} \\
&< \delta
\end{aligned} \tag{4.111}$$

is a contradiction to the assumption that $\|\boldsymbol{x}(t^*)\| = \delta$. Thus, Claim 4.3.1 is true; namely, $\|\boldsymbol{x}(t)\| \leq \delta$ for all $t \in (\tau_k, \tau_{k+1}]$ whenever $\boldsymbol{x}(\tau_k^+) \leq \delta_1$. Then from (4.106) we have

$$\|\boldsymbol{g}(\boldsymbol{x}(t))\| \leq \|\boldsymbol{x}(t)\| \tag{4.112}$$

for all $t \in (\tau_k, \tau_{k+1}]$ and $\|\boldsymbol{x}(\tau_k^+)\| \leq \delta_1$.

Let us choose the following Lyapunov function

$$V(t, \boldsymbol{x}) = \boldsymbol{x}(t)^\top P \boldsymbol{x}(t) \tag{4.113}$$

then for $(t, \boldsymbol{x}) \in \mathfrak{G}$ we have

$$\begin{aligned} \dot{V}(t, \boldsymbol{x}) &= \boldsymbol{x}^\top (A^\top P + PA)\boldsymbol{x} + [\boldsymbol{g}(\boldsymbol{x})^\top P\boldsymbol{x} + \boldsymbol{x}^\top P\boldsymbol{g}(\boldsymbol{x})] \\ &= -\boldsymbol{x}^\top Q\boldsymbol{x} + [\boldsymbol{g}(\boldsymbol{x})^\top P\boldsymbol{x} + \boldsymbol{x}^\top P\boldsymbol{g}(\boldsymbol{x})] \qquad \Leftarrow (4.102) \\ &\leq \left(-\lambda_3 + 2\sqrt{\frac{\lambda_2}{\lambda_1}}\right) V(t, \boldsymbol{x}), \qquad (t, \boldsymbol{x}) \in \mathfrak{G}, \end{aligned} \tag{4.114}$$

whenever $\|\boldsymbol{x}(\tau_k^+)\| \leq \delta_1$. By choosing

$$\varrho(w) = \left(\lambda_3 - 2\sqrt{\frac{\lambda_2}{\lambda_1}}\right) w$$

the conclusion is straightforward from Corollary 4.1.1. ■

For $t \in (\tau_k, \tau_{k+1}]$, the solution of (4.99) is given by

$$\boldsymbol{x}(t) = e^{A(t-\tau_k)}\boldsymbol{x}(\tau_k^+) + \int_{\tau_k}^{t} e^{A(t-s)}\boldsymbol{g}(\boldsymbol{x}(s))ds. \tag{4.115}$$

When $t = \tau_{k+1}$, we have

$$\boldsymbol{x}(\tau_{k+1}) = e^{A(\tau_{k+1}-\tau_k)}\boldsymbol{x}(\tau_k^+) + \int_{\tau_k}^{\tau_{k+1}} e^{A(\tau_{k+1}-s)}\boldsymbol{g}(\boldsymbol{x}(s))ds. \tag{4.116}$$

Let us use the following notation:

$$\xi_k \triangleq \int_{\tau_k}^{\tau_{k+1}} e^{A(\tau_{k+1}-s)}\boldsymbol{g}(\boldsymbol{x}(s))ds. \tag{4.117}$$

For any given $\epsilon > 0$, we choose $\epsilon_1 > 0$ such that

$$\epsilon = \epsilon_1 e^{(2\|A\|+1)(\tau_{k+1}-\tau_k)}. \tag{4.118}$$

From (4.105) we know that there is a $\delta_2 > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{x})\| \leq \epsilon_1 \|\boldsymbol{x}\|, \qquad \text{provided } \|\boldsymbol{x}\| < \delta_2. \tag{4.119}$$

Let us choose

$$\delta_3 = \min\left(\delta_1, \frac{\delta_2}{e^{(\|A\|+1)(\tau_{k+1}-\tau_k)}}\right) \tag{4.120}$$

then in case of $\|\boldsymbol{x}\| \leq \delta_3$, from (4.110) we have

$$\begin{aligned}
\|\boldsymbol{x}(t)\| &\leq \|\boldsymbol{x}(\tau_k)^+\| e^{(\|A\|+1)(\tau_{k+1}-\tau_k)} \qquad \Leftarrow (4.110)\\
&\leq \delta_3 e^{(\|A\|+1)(\tau_{k+1}-\tau_k)}\\
&\leq \delta_2, \qquad t \in (\tau_k, \tau_{k+1}]. \qquad \Leftarrow (4.120)
\end{aligned} \tag{4.121}$$

From (4.110), (4.119) and (4.121) we know that

$$\begin{aligned}
\|\boldsymbol{g}(\boldsymbol{x}(t))\| &\leq \epsilon_1 \|\boldsymbol{x}(t)\| \qquad \Leftarrow (4.119)\\
&\leq \epsilon_1 \|\boldsymbol{x}(\tau_k^+)\| e^{(\|A\|+1)(\tau_{k+1}-\tau_k)}, \qquad \Leftarrow (4.110)\\
&\quad \text{provided } \|\boldsymbol{x}(\tau_k^+)\| \leq \delta_3, \quad t \in (\tau_k, \tau_{k+1}].
\end{aligned} \tag{4.122}$$

We then have

$$\begin{aligned}
\|\xi_k\| &\leq \int_{\tau_k}^{\tau_{k+1}} e^{\|A\|(\tau_{k+1}-\tau_k)} \|\boldsymbol{g}(\boldsymbol{x}(s))\| ds \qquad \Leftarrow (4.117)\\
&\leq e^{\|A\|(\tau_{k+1}-\tau_k)} \epsilon_1 \|\boldsymbol{x}(\tau_k^+)\| e^{(\|A\|+1)(\tau_{k+1}-\tau_k)} \qquad \Leftarrow (4.118)\\
&= \epsilon \|\boldsymbol{x}(\tau_k^+)\|, \qquad \text{provided } \|\boldsymbol{x}(\tau_k^+)\| \leq \delta_3.
\end{aligned} \tag{4.123}$$

Thus for any given $\epsilon > 0$, there is a $\delta_3 > 0$, $\delta_3 \leq \delta_1$, such that $\|\xi_k\| \leq \epsilon \|\boldsymbol{x}(\tau_k^+)\|$ whenever $\|\boldsymbol{x}(\tau_k^+)\| < \delta_3$.

Then from (4.116) we have for any given $\epsilon > 0$

$$\begin{aligned}
\|\boldsymbol{x}(\tau_{k+1})\| &\leq e^{\|A\|(\tau_{k+1}-\tau_k)} \|\boldsymbol{x}(\tau_k^+)\| + \|\xi_k\| \qquad \Leftarrow (4.116)\&(4.117)\\
&\leq \left(e^{\|A\|(\tau_{k+1}-\tau_k)} + \epsilon\right) \|\boldsymbol{x}(\tau_k^+)\| \qquad \Leftarrow (4.123)\\
&\quad \text{provided } \|\boldsymbol{x}(\tau_k^+)\| \leq \delta_3.
\end{aligned} \tag{4.124}$$

If the impulsive control law is given by $U(k, \boldsymbol{x}) = B_k \boldsymbol{x}$, then we immediately have the following theorem:

Theorem 4.3.2. *Assume that*

1.
$$0 < \gimel_1 = \inf_{k \in \mathbb{N}} (\tau_{k+1} - \tau_k) \leq \sup_{k \in \mathbb{N}} (\tau_{k+1} - \tau_k) = \gimel_2 < \infty;$$

2.
$$\|(I + B_k) e^{A(\tau_{k+1}-\tau_k)}\| \leq \gimel_3 < 1, \qquad k \in \mathbb{N}.$$

Then the origin $\boldsymbol{x} = 0$ of system (4.99) is asymptotically stable. ⊠

If

$$\Lambda = \lim_{k \to \infty} (I + B_k) e^{A(\tau_{k+1}-\tau_k)}$$

exists, then we have the following theorem

Theorem 4.3.3. *Assume that assumption (1) of Theorem 4.3.2 is satisfied and the spectral radius of Λ, $\rho(\Lambda) < 1$, then the origin $\boldsymbol{x} = 0$ of system (4.99) is asymptotically stable.* ⊠

4.4 Linear Approximation Methods

Consider the following impulsive differential system

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), & t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x} + U(k, \boldsymbol{x}), \, t = \tau_k, & k = 1, 2, \cdots, \end{cases} \tag{4.125}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{g} : \mathbb{R}_+ \times \mathcal{S}_\rho \to \mathbb{R}^n$, $\rho > 0$, $B_k \in \mathbb{R}^{n\times n}$, $U : \mathbb{N} \times \mathcal{S}_\rho \to \mathbb{R}^n$. Let $\Phi(t, s)$ be the fundamental matrix of the following ordinary differential equation:

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}. \tag{4.126}$$

Then the stability of system (4.125) based on linear approximation is given by the following theorem.

Theorem 4.4.1. *Assume that*

1. $A \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^{n\times n}]$;
2. $\|\Phi(t, s)\| \leq \phi(t)\psi(s)$ *for* $\tau_{k-1} < s \leq t \leq \tau_k, k \in \mathbb{N}$, *where* $\phi, \psi \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ *and* $\phi(t) > 0$, $\psi(t) > 0$, $\phi(\tau_k^+) > 0$, $\psi(\tau_k^+) > 0$ *for* $t \in \mathbb{R}_+$, $k \in \mathbb{N}$;
3. *the function* $\boldsymbol{g}(t, \boldsymbol{x})$ *is continuous and locally Lipschitz continuous with respect to* $\boldsymbol{x}$ *in the sets* $(\tau_{k-1}, \tau_k] \times \mathcal{S}_\rho$, $k \in \mathbb{N}$, *and for each* k *and* $\boldsymbol{y} \in \mathcal{S}_\rho$ *the limit of* $\boldsymbol{g}(t, \boldsymbol{x})$ *exists as* $(t, \boldsymbol{x}) \to (\tau_k, \boldsymbol{y})$, $t > \tau_k$;
4. *in the domain* $\mathbb{R}_+ \times \mathcal{S}_\rho$, *the following inequality holds:* $\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq \alpha(t)\|\boldsymbol{x}\|^m$, *where* $m \geq 1$ *is a constant and the function* $\alpha \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ *is nonnegative;*
5. $\|I + B_k\| \leq \gamma_k$, $\gamma_k \geq 0$ *is a constant and* I *is the identity matrix;*
6. *the functions* $U(k, \boldsymbol{x})$ *are continuous in* $\mathcal{S}_\rho$ *and*
$$\|U(k, \boldsymbol{x})\| \leq \rho_k\|\boldsymbol{x}\|, \quad \boldsymbol{x} \in \mathcal{S}_\rho, \quad k \in \mathbb{N}$$
where $\rho_k \geq 0$ *are constants.*
7. *for any* $t_0 \geq 0$ *there is a* $N = N(t_0)$ *such that*
$$\phi(t) \prod_{t_0<\tau_k<t} r_k < N, \quad \textit{for } t > t_0,$$
where $r_k = (\gamma_k + \rho_k)\phi(\tau_k)\psi(\tau_k^+)$
8.
$$D(0, \infty) \triangleq \int_0^\infty \left(\prod_{0<\tau_k<s} r_k\right)^{m-1} \phi^m(s)\psi(s)\alpha(s)ds < \infty; \tag{4.127}$$

9. *there is a* $N > 0$ *such that*

$$\phi(t)\psi(t_0^+) \prod_{t_0<\tau_k<t} r_k < N, \qquad \textit{for } 0 \le t_0 \le t < \infty,$$

$$\psi^{m-1}(t_0^+)D(t_0,\infty) < N, \qquad \textit{for } t_0 \ge 0;$$

10. *for any* $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$ *there is a* $\sigma = \sigma(\epsilon, t_0)$ *such that*

$$\phi(t) \prod_{0<\tau_k<t} r_k < \epsilon, \ \textit{for} \ \ t \ge t_0 + \sigma; \tag{4.128}$$

11. *for any* $\epsilon > 0$ *there is a* $\sigma = \sigma(\epsilon)$ *such that for each* $t_0 \in \mathbb{R}_+$ *we have*

$$\phi(t)\psi(t_0^+) \prod_{0<\tau_k<t} r_k < \epsilon, \ \textit{for} \ \ t \ge t_0 + \sigma.$$

Let us assume that $m > 1$*, then the solution* $\boldsymbol{x} = 0$ *of system (4.125) is*

1. *stable if assumptions 1-8 hold;*
2. *uniformly stable if assumptions 1-6 and 9 hold;*
3. *asymptotically stable if assumptions 1-6, 8 and 10 hold;*
4. *uniformly asymptotically stable if assumptions 1-6, 9 and 11 hold.*

⊠

Proof. See the proofs of Theorems 8.1-8.4 of [2]. ■

Theorem 4.4.2. *Given the following function:*

$$M(t_0,t) = \phi(t)\psi(t_0^+) \left[\prod_{0<\tau_k<t} (\gamma_k + \rho_k)\phi(\tau_k)\psi(\tau_k^+) \right] \exp\left(\int_0^t \phi(s)\psi(s)ds \right)$$

and assume that the assumptions 1-6 of Theorem 4.4.1 hold and $m = 1$*, then the solution* $\boldsymbol{x} = 0$ *of system (4.125) is*

1. *stable if for any* $t_0 \in \mathbb{R}_+$ *there is a* $N > 0$ *such that* $M(t_0,t) \le N$ *for* $t \ge t_0 \ge 0$*;*
2. *asymptotically stable if for any* $t_0 \in \mathbb{R}_+$ *and* $\epsilon > 0$ *there is a* $\sigma > 0$ *such that* $M(t_0,t) \le \epsilon$ *for* $t \ge t_0 + \sigma$*;*
3. *uniformly stable if there is a* $N > 0$ *such that* $M(t_0,t) \le N$ *for* $t \ge t_0 \ge 0$*;*
4. *uniformly asymptotically stable if there is a* $N > 0$ *such that* $M(t_0,t) \le N$ *for* $t \ge t_0 \ge 0$ *and for any* $\epsilon > 0$ *there is a* $\sigma > 0$ *such that* $M(t_0,t) \le \epsilon$ *for any* $t_0 \in \mathbb{R}_+$ *and* $t \ge t_0 + \sigma$.

⊠

Proof. See the proof of Theorems 8.5 of [2]. ■

Let us then consider the following special case:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + P(t)\boldsymbol{x}, \ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x} + B_k\boldsymbol{x}, \ t = \tau_k, \quad k = 1, 2, \cdots, \end{cases} \qquad (4.129)}$$

where $A \in \mathbb{R}^{n\times n}$, $P(t) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$. Let us also consider the following reference system

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad t \neq \tau_k, \\ \Delta\boldsymbol{x} = B\boldsymbol{x}, \ t = \tau_k, \quad k = 1, 2, \cdots . \end{cases} \tag{4.130}$$

We then have the following theorem.

Theorem 4.4.3. *If the solutions of the reference system (4.130) are stable, $I + B$ is nonsingular*

$$\int_{t_0}^{\infty} \|P(t)\|dt < \infty \ \textit{and} \prod_{\tau_i > t_0} (1 + \|B_i\|) < \infty \tag{4.131}$$

then the solutions of system (4.129) are also stable. ⊠

Proof. The state transition matrix of system (4.130) is given by

$$\Psi(t, t_0) = e^{A(t-\tau_i)} \prod_{t_0 < \tau_j < \tau_i} (I + B)e^{A(\tau_j - \tau_{j-1})},$$
$$\tau_0 = t_0, \quad t \in (\tau_i, \tau_{i+1}]. \tag{4.132}$$

Since the matrix $I + B$ is nonsingular, $\Psi(t, t_0)$ is also nonsingular and we have

$$\Psi(t, t_0)\Psi^{-1}(s, t_0) = e^{A(t-\tau_i)} \prod_{t_0 < \tau_j < \tau_i} (I + B)e^{A(\tau_j - \tau_{j-1})}(I + B)e^{A(\tau_{k+1} - s)},$$
$$\tau_i < t \leq \tau_{i+1}, \quad \tau_k < s < \tau_{k+1}, \quad k < i. \tag{4.133}$$

Since the solutions of system (4.130) are stable, it follows from (4.133) that there is a number $K > 0$ such that

$$\|\Psi(t, t_0)\| \leq K, \quad \|\Psi(t, t_0)\Psi^{-1}(s, t_0)\| \leq K, \quad t_0 \leq s \leq t. \tag{4.134}$$

With the initial condition $\boldsymbol{x}(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$, any a solution of system (4.129) is given by

$$\boldsymbol{x}(t, \boldsymbol{x}_0) = \Psi(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Psi(t, s)P(s)\boldsymbol{x}(s, \boldsymbol{x}_0)ds$$
$$+ \sum_{t_0 < \tau_i < t} \Psi(t, \tau_i)B_i\boldsymbol{x}(\tau_i, \boldsymbol{x}_0). \tag{4.135}$$

Let $\boldsymbol{x}(t, \boldsymbol{x}_0)$ and $\boldsymbol{x}(t, \boldsymbol{y}_0)$ be any two solutions of (4.129) and from (4.134) we have

$$\begin{aligned}\|\boldsymbol{x}(t, \boldsymbol{x}_0) - \boldsymbol{x}(t, \boldsymbol{y}_0)\| \leq K\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| \\ + \int_{t_0}^{t} K\|P(s)\|\|\boldsymbol{x}(s, \boldsymbol{x}_0) - \boldsymbol{x}(s, \boldsymbol{y}_0)\|ds \\ + \sum_{t_0<\tau_i<t} K\|B_i\|\|\boldsymbol{x}(\tau_i, \boldsymbol{x}_0) - \boldsymbol{x}(\tau_i, \boldsymbol{y}_0)\|.\end{aligned}$$

From Lemma 1.7.1 it follows

$$\begin{aligned}&\|\boldsymbol{x}(t, \boldsymbol{x}_0) - \boldsymbol{x}(t, \boldsymbol{y}_0)\| \\ &\leq K \prod_{t_0<\tau_i<t} (1 + K\|B_i\|) \exp\left(\int_{t_0}^{t} K\|P(s)\|ds\right) \|\boldsymbol{x}_0 - \boldsymbol{y}_0\|, \qquad t \geq t_0.\end{aligned}$$

Since $1 + K\|B_i\| \leq (1 + \|B_i\|)^K$ and from (4.131) we know that $\prod_{t_0<\tau_i<t}(1 + K\|B_i\|)$ converges. Thus we know that the solutions of system (4.129) is stable.

■

Consider the following impulsive differential system

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x}, \qquad\qquad t = \tau_k, \quad k = 1, 2, \cdots \end{cases} \tag{4.136}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{g} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Let $\Phi(t, s)$ be the fundamental matrix of the following ODE:

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}. \tag{4.137}$$

Theorem 4.4.4. *Let us assume that*

1. $A \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^{n\times n}]$;
2. $\|\Phi(t, s)\| \leq \phi(t)\psi(s)$ *for* $\tau_{k-1} < s \leq t \leq \tau_k, k \in \mathbb{N}$, *where* $\phi, \psi \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ *and* $\phi(t) > 0$, $\psi(t) > 0$, $\phi(\tau_k^+) > 0$, $\psi(\tau_k^+) > 0$ *for* $t \in \mathbb{R}_+$, $k \in \mathbb{N}$;
3. *The function* $\boldsymbol{g}(t, \boldsymbol{x})$ *is continuous and locally Lipschitz continuous with respect to* $\boldsymbol{x}$ *in the sets* $(\tau_{k-1}, \tau_k] \times \mathbb{R}^n$, $k \in \mathbb{N}$, *and for each* k *and* $\boldsymbol{y} \in \mathbb{R}^n$ *the limit of* $\boldsymbol{g}(t, \boldsymbol{x})$ *exists as* $(t, \boldsymbol{x}) \to (\tau_k, \boldsymbol{y})$, $t > \tau_k$;
4. *In the domain* $\mathbb{R}_+ \times \mathbb{R}^n$, *the following inequality holds:* $\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq \alpha(t)\|\boldsymbol{x}\|^m$, *where* $m > 1$ *is a constant and the function* $\alpha \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ *is nonnegative;*
5. $\|I + B_k\| \leq \gamma_k$, *where* $\gamma_k \geq 0$ *is a constant;*

6.

$$D(0,\infty) \triangleq \int_0^\infty \left(\prod_{0<\tau_k<s} r_k \right)^{m-1} \phi^m(s)\psi(s)\alpha(s)ds < \infty \quad (4.138)$$

where $r_k = \gamma_k \phi(\tau_k)\psi(\tau_k^+)$*;*

7. For any $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$ *there exists* $\sigma = \sigma(\epsilon, t_0)$ *such that*

$$\phi(t) \prod_{0<\tau_k<s} r_k < \epsilon, \ \textit{for } t \geq t_0 + \sigma. \quad (4.139)$$

Then the solution $\boldsymbol{x} = 0$ *of system (4.136) is asymptotically stable.* ⊠

Proof. It is straightforward from Theorem 4.4.1. ■

4.5 Stability of Sets

In this section let us study the impulsive control problem of controlling the plant to a target set instead of a target point. This kind of control problem is very common when the plant is chaotic because a chaotic attractor remains within a certain region. The design of this kind of impulsive controller is based on the stability of sets[2, 8] of impulsive control systems. Let $\Xi \subseteq \mathbb{R}^n$ and assume that there is a set $\$ \subset \mathbb{R}_+ \times \Xi$ in the phase space of the following impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta\boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \quad k \in \mathbb{N}, \quad t \in \mathbb{R}_+ \end{aligned}} \quad (4.140)$$

where

1. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \Xi, \mathbb{R}^n]$ and there is a constant $L_1 > 0$ such that $\|\boldsymbol{f}(t,\boldsymbol{x}) - \boldsymbol{f}(t,\boldsymbol{y})\| \leq L_1\|\boldsymbol{x} - \boldsymbol{y}\|$ for $t \in \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \Xi$. There is a $K \in \mathbb{R}_+$ such that $\|f(t,\boldsymbol{x})\| \leq K$ for $(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \Xi$.
2. $\tau_k \in C[\Xi, \mathbb{R}_+], k \in \mathbb{N}$ satisfy

$$0 = \tau_0 < \tau_1(\boldsymbol{x}) < \tau_2(\boldsymbol{x}) < \cdots, \quad \lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty \text{ uniformly on } \boldsymbol{x} \in \Xi$$

and

$$\inf_{\Xi} \tau_{k+1}(\boldsymbol{x}) - \sup_{\Xi} \tau_k(\boldsymbol{x}) \geq \theta > 0.$$

3. $U : \mathbb{N} \times \Xi \to \mathbb{R}^n$ satisfies

$$\|U(k,\boldsymbol{x}) - U(k,\boldsymbol{y})\| \leq L_2\|\boldsymbol{x} - \boldsymbol{y}\|, \quad \boldsymbol{x}, \boldsymbol{y} \in \Xi, \quad k \in \mathbb{N},$$

and $\boldsymbol{x} + U(k,\boldsymbol{x}) \in \Xi$ for all $\boldsymbol{x} \in \Xi$ and $k \in \mathbb{N}$.

We assume that there is no beating phenomenon. Let us define $\Gamma_k \triangleq (\tau_{k-1}(\boldsymbol{x}), \tau_k(\boldsymbol{x})) \times \Xi$, $k \in \mathbb{N}$ and

$$\Gamma \triangleq \bigcup_{k=1}^{\infty} \Gamma_k.$$

$\$(t) \triangleq \{\boldsymbol{x} \in \Xi |\ (t, \boldsymbol{x}) \in \$\}$, $\$_k \triangleq \{\boldsymbol{x} \in \Xi |\ (\tau_k, \boldsymbol{x}) \in [\$ \cap \Gamma_{k+1}]^c\}$,

$$\$(t^+) \triangleq \begin{cases} \$(t),\ t \neq \tau_k, \\ \$_k, \quad t = \tau_k. \end{cases} \tag{4.141}$$

Let there be such a $\rho > 0$ that for all $t \in \mathbb{R}_+$ we have $[\$(t, \rho)]^c \subset \Xi$ and let $d(\boldsymbol{x}, \$)$ be a distance between a point $\boldsymbol{x} \in \mathbb{R}^n$ and the set $\$$ defined by

$$d(\boldsymbol{x}, \$) \triangleq \inf_{\boldsymbol{y} \in \$} \|\boldsymbol{x} - \boldsymbol{y}\|$$

and satisfying

$$\lim_{t \to \tau_k^+} d(\boldsymbol{x}, \$(t)) = d(\boldsymbol{x}, \$_k), \quad k \in \mathbb{N}, \quad \boldsymbol{x} \in \$(\tau_k^+, \rho). \tag{4.142}$$

then we define the following notations of ξ-neighborhoods of $\$(t)$:

$$\begin{aligned} \$(t, \xi) &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n |\ d(\boldsymbol{x}, \$(t)) < \xi\}, \\ \$(t^+, \xi) &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n |\ d(\boldsymbol{x}, \$(t^+)) < \xi\}. \end{aligned} \tag{4.143}$$

We assume that $\$(t) \neq \emptyset$ for all $t \in \mathbb{R}_+$ and $\$_k \neq \emptyset$ for all $k \in \mathbb{N}$. We also assume that for any compact subset S of $\mathbb{R}_+ \times \Xi$ there is a constant $L_3 > 0$ which is dependent on S such that if $(t_1, \boldsymbol{x})$, $(t_2, \boldsymbol{x}) \in S$, then we have

$$|d(\boldsymbol{x}, \$(t_1)) - d(\boldsymbol{x}, \$(t_2))| \leq L_3 \|t_1 - t_2\|.$$

Suppose that there is a $\varrho > 0$ such that the set $\{\boldsymbol{x} \in \mathbb{R}^n \mid d(\boldsymbol{x}, \$(t)) \leq \varrho\}$ is a subset of Ξ and any a solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of impulsive control system (4.140) satisfies $d(\boldsymbol{x}(t), \$(t)) \leq \varrho$ for all $t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0)$.

Definition 4.5.1. $V : \mathbb{R}_+ \times \Xi \to \mathbb{R}$ *belongs to the class* $\mathcal{V}_3$ *if*

1. *V is continuous on Γ and locally Lipschitz continuous with respect to each $\boldsymbol{x}$ in $\Gamma_k, k \in \mathbb{N}$;*
2. *$V(t, \boldsymbol{x}) = 0$ for $(t, \boldsymbol{x}) \in \$$ and $V(t, \boldsymbol{x}) > 0$ for $(t, \boldsymbol{x}) \in (\mathbb{R}_+ \times \Xi) \setminus \$$;*
3. *for any $\boldsymbol{x} \in \Xi$ and $k \in \mathbb{N}$ the following limits exist:*

$$\begin{aligned} \lim_{(t,\boldsymbol{y}) \to (\tau_k^+, \boldsymbol{x})} V(t, \boldsymbol{y}) &= V(\tau_k^+, \boldsymbol{x}), \\ \lim_{(t,\boldsymbol{y}) \to (\tau_k^-, \boldsymbol{x})} V(t, \boldsymbol{y}) &= V(\tau_k^-, \boldsymbol{x}) = V(\tau_k, \boldsymbol{x}). \end{aligned} \tag{4.144}$$

⊠

4.5.1 Stability

In this section all results are presented under assumption $\Xi \subset \mathbb{R}^n$. We first present some definitions of different kinds of stability of sets.

Definition 4.5.2. *Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of (4.140) such that $\boldsymbol{x}(t_0^+, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$, then the set \$ is*

SS1 *a stable set of system (4.140) if for any $\delta > 0$, $t_0 \in \mathbb{R}_+$ and $\eta > 0$ there is a $\varepsilon = \varepsilon(t_0, \delta, \eta)$ such that $\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \varepsilon)$ implies $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta)$ for any $t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0)$.*

SS2 *a t-uniformly stable set of system (4.140) if the ε in* SS1 *is independent of t_0.*

SS3 *an η-uniformly stable set of system (4.140) if the ε in* SS1 *is independent of η.*

SS4 *a uniformly stable set of system (4.140) if the ε in* SS1 *is independent of t_0 and η.*

⊠

Definition 4.5.3. *Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of (4.140) such that $\boldsymbol{x}(t_0^+, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$, then the set \$ is*

SA1 *an attractive set of system (4.140) if for each $\eta > 0$, $t_0 \in \mathbb{R}_+$ there is a $\mu > 0$ and for each $\delta > 0$, $\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \mu)$ there is a $\phi > 0$ that satisfies $t_0 + \phi \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0)$ such that*

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t \geq t_0 + \phi, t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0).$$

SA2 *a t-uniformly attractive set of system (4.140) if for each $\eta > 0$ there is a $\mu > 0$, for each $\delta > 0$ there is a $\phi > 0$, such that for each $t_0 \in \mathbb{R}_+$*

$$\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \mu), \quad t_0 + \phi \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0),$$

implies

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t \geq t_0 + \phi, t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0).$$

SA3 *an η-uniformly attractive set of system (4.140) if for each $t_0 \in \mathbb{R}_+$ there is a $\mu > 0$, for each $\delta > 0$ there is a $\phi > 0$, such that for each $\eta > 0$*

$$\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \mu), \quad t_0 + \phi \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0),$$

implies

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t \geq t_0 + \phi, t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0).$$

SA4 *a uniformly attractive set of system (4.140) if given a* $\mu > 0$ *and for each* $\delta > 0$ *there is a* $\phi > 0$*, such that for each* $\eta > 0$, $t_0 \in \mathbb{R}_+$

$$\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \mu), \quad t_0 + \phi \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0),$$

implies

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t \geq t_0 + \phi, t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0).$$

⊠

Definition 4.5.4. $\$$ *is*

1. *an asymptotically stable set of system (4.140) if it is* SS1 *and* SA1.
2. *a t-uniformly asymptotically stable set of system (4.140) if it is* SS2 *and* SA2.
3. *an* η*-uniformly asymptotically stable set of system (4.140) if it is* SS3 *and* SA3.
4. *a uniformly asymptotically stable set of system (4.140) if it is* SS4 *and* SA4.

⊠

Theorem 4.5.1. *Assume that* $\alpha \in \mathcal{K}$ *and* $V \in \mathcal{V}_3$ *and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.145}$$

$$D^+ V(t, \boldsymbol{x}) \leq 0 \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.146}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi. \tag{4.147}$$

Then $\$$ *is a stable set of system (4.140).* ⊠

Proof. Let $\delta, \eta > 0$ and given $t_0 \in \mathbb{R}_+$, $\boldsymbol{x}_0 \in \Xi$, then it follows from $V \in \mathcal{V}_3$ that $V(t_0, \boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \$(t_0)$. Therefore, there is a $\varepsilon = \varepsilon(t_0, \delta, \eta) > 0$ such that, if $\boldsymbol{x} \in \overline{\mathcal{S}_\eta} \cap \$(t_0^+, \varepsilon) \cap \Xi$, then

$$V(t_0^+, \boldsymbol{x}) < \alpha(\delta). \tag{4.148}$$

Choose $\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0^+, \varepsilon) \cap \Xi$, then it follows from (4.145), (4.146) and (4.147) that

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \\ &\leq V(t_0^+, \boldsymbol{x}_0) < \alpha(\delta), \end{aligned} \tag{4.149}$$

from which we have

$$d(\boldsymbol{x}(t), \$(t)) < \delta. \tag{4.150}$$

Thus, $\boldsymbol{x}(t) \in \$(t, \delta)$ for all $t > t_0$. This proves that $\$$ is a stable set of system (4.140). ■

Theorem 4.5.2. *Assume that* $\alpha \in \mathcal{K}$, $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\beta(t, \cdot) \in \mathcal{K}$ *for any fixed* $t \in \mathbb{R}_+$, $V \in \mathcal{V}_3$ *and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.151}$$

$$\beta(t, d(\boldsymbol{x}, \$(t))) \ge V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.152}$$

$$D^+V(t, \boldsymbol{x}) \le 0 \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.153}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi. \tag{4.154}$$

Then \$ *is an* η*-uniformly stable set of system (4.140).* ⊠

Proof. For any $\delta > 0$ and $t_0 \in \mathbb{R}_+$ let us choose a $\varepsilon = \varepsilon(t_0, \delta) > 0$ such that $\beta(t_0, \varepsilon) < \alpha(\delta)$. Let us choose $\eta > 0$ and $\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0^+, \varepsilon) \cap \Xi$, then it follows from (4.151), (4.153), (4.154) and (4.152) that

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\le V(t, \boldsymbol{x}(t)) \\ &\le V(t_0^+, \boldsymbol{x}_0) \\ &\le \beta(t_0, d(\boldsymbol{x}_0, \$(t_0))) \\ &\le \beta(t_0, \varepsilon) < \alpha(\delta), \quad t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0). \end{aligned} \tag{4.155}$$

Therefore we know that $\mathfrak{J}^+(t_0, \boldsymbol{x}_0) = (t_0, \infty)$ and $\boldsymbol{x}(t) \in \$(t, \delta)$ for all $t > t_0$. This proves that \$ is an η-uniformly stable set of system (4.140). ■

Similarly, we have the following two theorems.

Theorem 4.5.3. *Assume that* $\alpha \in \mathcal{K}$, $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\beta(w, \cdot) \in \mathcal{K}$ *for any fixed* $w \in \mathbb{R}_+$, $V \in \mathcal{V}_3$ *and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.156}$$

$$\beta(\|\boldsymbol{x}\|, d(\boldsymbol{x}, \$(t))) \ge V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.157}$$

$$D^+V(t, \boldsymbol{x}) \le 0 \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.158}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi. \tag{4.159}$$

Then \$ *is a* t*-uniformly stable set of system (4.140).* ⊠

Theorem 4.5.4. *Assume that* $\alpha, \beta \in \mathcal{K}$, $V \in \mathcal{V}_3$ *and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.160}$$

$$\beta(d(\boldsymbol{x}, \$(t))) \ge V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.161}$$

$$D^+V(t, \boldsymbol{x}) \le 0 \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.162}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi. \tag{4.163}$$

Then \$ *is a uniformly stable set of system (4.140).* ⊠

Theorem 4.5.5. *Assume that* $\alpha, \beta, \kappa \in \mathcal{K}$ *and* $V, V_1 \in \mathcal{V}_3$ *and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \ \textit{for} \ (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.164}$$

$$\beta(d(\boldsymbol{x}, \$(t))) \le V_1(t, \boldsymbol{x}) \ \textit{for} \ (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.165}$$

$$D^+V(t, \boldsymbol{x}) \le -\kappa(V_1(t, \boldsymbol{x})) \ \textit{for} \ (t, \boldsymbol{x}) \in \Gamma, \tag{4.166}$$

$$\sup_{(t,\boldsymbol{x})\in\Gamma} D^+V_1(t, \boldsymbol{x}) \le K_1 < \infty, \tag{4.167}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \ \textit{for} \ k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \tag{4.168}$$

$$V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V_1(\tau_k, \boldsymbol{x}) \ \textit{for} \ k \in \mathbb{N}, \boldsymbol{x} \in \Xi. \tag{4.169}$$

Then \$ is an asymptotically stable set of system (4.140). ⊠

Proof. It follows from Theorem 4.5.1 that \$ is a stable set of system (4.140). Let us choose an $\eta > 0$ such that for all $t \in \mathbb{R}_+$ we have $\$(t, \eta) \subset \Xi$. It follows from (4.164) that

$$\Theta(t, \eta) \subset \$(t, \eta) \subset \Xi, \quad t \in \mathbb{R}_+ \tag{4.170}$$

where

$$\Theta(t, \eta) \triangleq \{\boldsymbol{x} \in \Xi \mid V(t^+, \boldsymbol{x}) \le \alpha(\eta)\}. \tag{4.171}$$

It follows from (4.166) and (4.168) that if $t_0 \in \mathbb{R}_+$ and $\boldsymbol{x}_0 \in \Theta(t_0, \eta)$, then $\boldsymbol{x}(t) \in \Theta(t, \eta)$ for $t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0)$. Therefore, it follows from (4.170) that $\boldsymbol{x}(t)$ will remain in the interior of Ξ for $t > t_0$ which implies that $\mathfrak{J}^+(t_0, \boldsymbol{x}_0) = (t_0, \infty)$. We then have the following claim:
Claim 4.5.5: Given $t_0 \in \mathbb{R}_+$ and $\boldsymbol{x}_0 \in \Theta(t_0, \eta)$, then

$$\lim_{t\to\infty} d(\boldsymbol{x}(t), \$(t)) = 0. \tag{4.172}$$

If Claim 4.5.5 is false, then there is a sequence $\{t_i\}_{i=1}^{\infty}$, $\lim_{i\to\infty} t_i = \infty$, such that for some $\sigma > 0$ we have $d(\boldsymbol{x}(t_i), \$(t_i)) \ge \sigma$ for all $i \in \mathbb{N}$. Then it follows from (4.165) that

$$V_1(t_i, \boldsymbol{x}(t_i)) \ge \beta(\sigma), \quad i \in \mathbb{N}. \tag{4.173}$$

Let us choose a subsequence $\{\tilde{t}_k\}_{k=1}^{\infty}$ of $\{t_i\}_{i=1}^{\infty}$ such that $\tilde{t}_k - \tilde{t}_{k-1} \ge \xi > 0$ and $\tilde{t}_k > t_0$ for each $k \in \mathbb{N}$. Let us choose $\varpi > 0$ such that

$$\varpi < \min\left(\xi, \frac{\beta(\sigma)}{2K_1}\right), \tag{4.174}$$

then it follows from (4.167), (4.173) and (4.174) that

$$\begin{aligned} V_1(t, \boldsymbol{x}(t)) &= V_1(\tilde{t}_k, \boldsymbol{x}(\tilde{t}_k)) + \int_{\tilde{t}_k}^{t} D^+V_1(s, \boldsymbol{x}(s))ds \\ &\ge \beta(\sigma) - K_1(\tilde{t}_k - t) \qquad \Leftarrow [(4.173)\&(4.167)] \\ &\ge \beta(\sigma) - K_1\varpi \\ &> \frac{\beta(\sigma)}{2}, \quad t \in [\tilde{t}_k - \varpi, \tilde{t}_k]. \qquad \Leftarrow (4.174) \end{aligned} \tag{4.175}$$

Therefore, it follows from (4.168), (4.166) and (4.175) that

$$\begin{aligned} V(\tilde{t}_k, \boldsymbol{x}(\tilde{t}_k)) &\le V(t_0^+, \boldsymbol{x}_0) + \int_{t_0}^{\tilde{t}_k} D^+V(s, \boldsymbol{x}(s))ds \qquad \Leftarrow [(4.168)\&(4.166)] \\ &\le V(t_0^+, \boldsymbol{x}_0) - \int_{t_0}^{\tilde{t}_k} \kappa(V_1(s, \boldsymbol{x}(s)))ds \qquad \Leftarrow (4.166) \\ &\le V(t_0^+, \boldsymbol{x}_0) - \sum_{i=1}^{k} \int_{\tilde{t}_k - \varpi}^{\tilde{t}_k} \kappa(V_1(s, \boldsymbol{x}(s)))ds \\ &\le V(t_0^+, \boldsymbol{x}_0) - \kappa\left(\frac{\beta(\sigma)}{2}\right) k\varpi \qquad \Leftarrow (4.175) \end{aligned} \tag{4.176}$$

from which we know that if k is big enough then $V(\tilde{t}_k, \boldsymbol{x}(\tilde{t}_k)) < 0$. This is a contradiction to (4.164). Hence, Claim 4.5.5 is true. Furthermore, since $\Theta(t_0, \eta)$ is a neighborhood of the origin that is in $\$(t_0, \eta)$, $\$$ is an attractive set of system (4.140). ■

Corollary 4.5.1. *Assume that the function $\boldsymbol{f}(t, \boldsymbol{x})$ is bounded in $\mathbb{R}_+ \times \Xi$ and Ξ is bounded, $\alpha, \kappa \in \mathcal{K}$, $V \in \mathcal{V}_3$ and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.177}$$
$$D^+V(t, \boldsymbol{x}) \le -\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.178}$$
$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \tag{4.179}$$
$$d(\boldsymbol{x} + U(k, \boldsymbol{x}), \$(t)) \le d(\boldsymbol{x}, \$(t)) \text{ for } k \in \mathbb{N}, (t, \boldsymbol{x}) \in \Sigma_k. \tag{4.180}$$

Then $\$$ is an asymptotically stable set of system (4.140). ⊠

Proof. Since $\boldsymbol{f}(t, \boldsymbol{x})$ is bounded in $\mathbb{R}_+ \times \Xi$ and Ξ is bounded, the derivative of $d(\boldsymbol{x}, \$(t))$ with respect to the solutions of system (4.140) is bounded. Let us set $V_1(t, \boldsymbol{x}) = d(\boldsymbol{x}, \$(t))$, then the conclusion follows from Theorem 4.5.5 immediately. ■

Corollary 4.5.2. *Assume that $\alpha, \kappa \in \mathcal{K}$ and $V \in \mathcal{V}_3$ and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.181}$$
$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \tag{4.182}$$
$$D^+V(t, \boldsymbol{x}) \le -\kappa(V(t, \boldsymbol{x})) \text{ for } (t, \boldsymbol{x}) \in \Gamma. \tag{4.183}$$

Then $\$$ is an asymptotically stable set of system (4.140). ⊠

Proof. Let us set $V_1(t, \boldsymbol{x}) = V(t, \boldsymbol{x})$, then the conclusion follows from Theorem 4.5.5 immediately. ■

Theorem 4.5.6. *Assume that $\alpha, \beta, \kappa \in \mathcal{K}$ and $V \in \mathcal{V}_3$ and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \tag{4.184}$$

$$D^+V(t, \boldsymbol{x}) \leq -\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma, \tag{4.185}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \tag{4.186}$$

$$V(t, \boldsymbol{x}) \leq \beta(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi. \tag{4.187}$$

Then \$ is a uniformly asymptotically stable set of system (4.140). ⊠

Proof. It follows from Theorem 4.5.4 that the set \$ is a uniformly stable set of system (4.140). Let us choose $\delta > 0$ and

$$\varepsilon = \varepsilon(\delta) < \beta^{-1}(\alpha(\delta)), \tag{4.188}$$

then it follows from (4.184), (4.185), (4.186) and (4.187) that for $t_0 \in \mathbb{R}_+$ and $\boldsymbol{x}_0 \in \$(t_0, \varepsilon)$ we have

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \leq V(t_0^+, \boldsymbol{x}_0) \\ &\leq \beta(d(\boldsymbol{x}_0, \$(t_0))) \leq \beta(\varepsilon(\delta)) \\ &< \alpha(\delta). \end{aligned} \tag{4.189}$$

Therefore we know that $d(\boldsymbol{x}(t), \$(t)) < \delta$; namely, $\boldsymbol{x}(t) \in \$(t, \delta)$ for $t > t_0$. Let us choose

$$\mu = \sup_{\delta \geq 0} \varepsilon(\delta), \phi = \phi(\delta) > \frac{\beta(\mu)}{\kappa(\varepsilon(\delta))}, \eta > 0,$$

and $\boldsymbol{x}_0 \in \overline{\mathcal{S}_\eta} \cap \$(t_0, \mu) \cap \Xi$. We then have the following claim:
Claim 4.5.6: There is such a $t_1 \in [t_0, t_0 + \phi]$ that $\boldsymbol{x}(t_1) \in \$(t_1, \varepsilon)$.

If Claim 4.5.6 is not true then we have

$$d(\boldsymbol{x}(t), \$(t)) \geq \varepsilon(\delta), \forall t \in [t_0, t_0 + \phi].$$

It follows from (4.185) that

$$\int_{t_0}^{t} D^+(s, \boldsymbol{x}(s)) ds \leq -(t - t_0)\kappa(\varepsilon(\delta)). \tag{4.190}$$

It follows from (4.186) that for $t \in (\tau_k, \tau_{k+1}], k \in \mathbb{N}$ we have

$$\begin{aligned} \int_{t_0}^{t} D^+(s, \boldsymbol{x}(s)) ds &= \sum_{i=1}^{k} \int_{\tau_{i-1}^+}^{\tau_i} D^+(s, \boldsymbol{x}(s)) ds + \int_{\tau_k^+}^{t} D^+(s, \boldsymbol{x}(s)) ds \\ &= \sum_{i=1}^{k} [V(\tau_i, \boldsymbol{x}(\tau_i)) - V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}^+))] \\ &\quad + V(t, \boldsymbol{x}(t)) - V(\tau_k^+, \boldsymbol{x}(\tau_k^+)) \\ &\geq \sum_{i=1}^{k} [V(\tau_i, \boldsymbol{x}(\tau_i)) - V(\tau_{i-1}, \boldsymbol{x}(\tau_{i-1}))] \\ &\quad + V(t, \boldsymbol{x}(t)) - V(\tau_k, \boldsymbol{x}(\tau_k)) \\ &= V(t, \boldsymbol{x}(t)) - V(t_0, \boldsymbol{x}_0). \end{aligned} \tag{4.191}$$

Then from (4.190) and (4.191) we have for $t \in [t_0, t_0 + \phi]$

$$V(t, \boldsymbol{x}(t)) \leq V(t_0, \boldsymbol{x}_0) - (t - t_0)\kappa(\varepsilon(\delta)) \tag{4.192}$$

from which we have

$$\begin{aligned} V(t_0 + \phi, \boldsymbol{x}(t_0 + \phi)) &\leq V(t_0, \boldsymbol{x}_0) - \phi\kappa(\varepsilon(\delta)) \\ &< \beta(\mu) - \frac{\beta(\mu)}{\kappa(\varepsilon(\delta))}\kappa(\varepsilon(\delta)) = 0 \end{aligned} \tag{4.193}$$

which is a contradiction to (4.184). Therefore Claim 4.5.6 is true. It follows from Claim 4.5.6 that for all $t \geq t_1$

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \\ &\leq V(t_1, \boldsymbol{x}(t_1)) \\ &\leq \beta(d(\boldsymbol{x}(t_1), \$(t_1))) \\ &\leq \beta(\varepsilon) < \alpha(\delta). \qquad \Leftarrow (4.188) \end{aligned} \tag{4.194}$$

Thus $d(\boldsymbol{x}(t), \$(t)) < \delta$; namely, $\boldsymbol{x}(t) \in \$(t, \delta)$ for all $t \geq t_1$ and $\$$ is a uniformly attractive set of system (4.140). ■

Similarly, we have the following two theorems.

Theorem 4.5.7. *Assume that $\alpha, \kappa \in \mathcal{K}$, $V \in \mathcal{V}_3$ and $\beta(t, \cdot) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and*

$$\begin{aligned} &\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \\ &D^+V(t, \boldsymbol{x}) \leq -\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma, \\ &V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \\ &V(t, \boldsymbol{x}) \leq \beta(t, d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi. \end{aligned} \tag{4.195}$$

Then $\$$ is an η-uniformly asymptotically stable set of system (4.140). ⊠

Theorem 4.5.8. *Assume that $\alpha, \kappa \in \mathcal{K}$, $V \in \mathcal{V}_3$ and $\beta(w, \cdot) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and*

$$\begin{aligned} &\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi, \\ &D^+V(t, \boldsymbol{x}) \leq -\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma, \\ &V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \Xi, \\ &V(t, \boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|, d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Xi. \end{aligned} \tag{4.196}$$

Then $\$$ is a t-uniformly asymptotically stable set of system (4.140). ⊠

4.5.2 Global Stability

In this section all results are presented under condition of $\Xi = \mathbb{R}^n$. We first provide some definitions related to global stability of sets.

Definition 4.5.5. *The control system (4.140) is*

SB1 $\$$*-equi-bounded if for each* $t_0 \in \mathbb{R}_+$, $\eta > 0$, $\varepsilon > 0$ *there is a* $\delta = \delta(t_0, \eta, \varepsilon)$ *such that*

$$\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \varepsilon)}$$

implies

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t > t_0.$$

SB2 t*-uniformly* $\$$*-bounded if* δ *in* SB1 *is independent of* t_0.
SB3 η*-uniformly* $\$$*-bounded if* δ *in* SB1 *is independent of* η.
SB4 *uniformly* $\$$*-bounded if* δ *in* SB1 *depends only on* ε.

⊠

Definition 4.5.6. $\$$ *is*

GA1 *Globally equi-attractive with respect to the control system (4.140) if for each* $t_0 \in \mathbb{R}_+$, $\eta > 0$, $\varepsilon > 0$, $\delta > 0$, *there is a* $\phi = \phi(t_0, \eta, \varepsilon, \delta) > 0$ *such that*

$$\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \varepsilon)}$$

implies

$$\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \in \$(t, \delta) \text{ for all } t \geq t_0 + \phi.$$

GA2 t*-uniformly globally attractive with respect to the control system (4.140) if* ϕ *in* GA1 *is independent of* t_0.
GA3 η*-uniformly globally attractive with respect to the control system (4.140) if* ϕ *in* GA1 *is independent of* η.
GA4 *uniformly globally attractive with respect to the control system (4.140) if* ϕ *in* GA1 *depends only on* δ *and* ε.

⊠

Definition 4.5.7. $\$$ *is*

1. *Globally equi-asymptotically stable with respect to the control system (4.140) if it is* SS1, SB1 *and* GA1.
2. t*-uniformly globally asymptotically stable with respect to the control system (4.140) if it is* SS2, SB2 *and* GA2.
3. η*-uniformly globally asymptotically stable with respect to the control system (4.140) if it is* SS3, SB3 *and* GA3.
4. *uniformly globally asymptotically stable with respect to the control system (4.140) if it is* SS4, SB4 *and* GA4.

⊠

We then provide some results on the global stability of set with respect to system (4.140) [9].

Theorem 4.5.9. *Assume that* $\alpha \in \mathcal{K}$ *satisfying*

$$\lim_{w \to \infty} \alpha(w) = \infty, \tag{4.197}$$

$V \in \mathcal{V}_3$, $\kappa > 0$ *is a constant and*

$$\alpha(d(\boldsymbol{x}, \$(t))) \le V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.198}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.199}$$

$$D^+V(t, \boldsymbol{x}) \le -\kappa V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \Gamma. \tag{4.200}$$

Then $\$$ *is a globally equi-asymptotically stable set with respect to system (4.140).* ⊠

Proof. Given $\eta > 0$, $\delta > 0$ and $t_0 \in \mathbb{R}_+$, then it follows from $V(t_0, \boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \$(t_0)$ that there is a $\varepsilon = \varepsilon(t_0, \eta, \delta)$ such that if $\boldsymbol{x} \in \mathcal{S}_\eta \cap \$(t_0, \varepsilon)$, then $V(t_0^+, \boldsymbol{x}) < \alpha(\delta)$. Let us choose $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \$(t_0, \varepsilon)$, then it follows from (4.198), (4.199) and (4.200) that

$$\alpha(d(\boldsymbol{x}(t), \$(t))) \le V(t, \boldsymbol{x}(t)) \le V(t_0^+, \boldsymbol{x}_0) < \alpha(\delta) \tag{4.201}$$

from which we know that $d(\boldsymbol{x}(t), \$(t)) < \delta$; namely, $\boldsymbol{x}(t) \in \$(t, \delta)$ for all $t > t_0$. This proves that the set $\$$ is stable with respect to (4.140).

Given $t_0 \in \mathbb{R}_+$, it follows from (4.199) and (4.200) that

$$V(t, \boldsymbol{x}(t)) \le V(t_0^+, \boldsymbol{x}_0) e^{-\kappa(t - t_0)}. \tag{4.202}$$

Given $\xi > 0$, $\eta > 0$, $\delta > 0$ and let

$$L_1(t_0, \xi, \eta) = \sup_{\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \xi)}} V(t_0^+, \boldsymbol{x}_0)$$

$$\phi = \phi(t_0, \eta, \xi, \delta) > \frac{1}{\kappa} \ln \frac{L_1(t_0, \xi, \eta)}{\alpha(\delta)}. \tag{4.203}$$

It follows from (4.198), (4.202) and (4.203) that for $t \ge t_0 + \phi$ we have

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\le V(t, \boldsymbol{x}(t)) \\ &\le V(t_0^+, \boldsymbol{x}_0) e^{-\kappa(t - t_0)} < \alpha(\delta) \end{aligned} \tag{4.204}$$

which proves that $\boldsymbol{x}(t) \in \$(t, \delta)$; namely, the set $\$$ is globally equi-attractive with respect to (4.140).

Given $t_0 \in \mathbb{R}_+, \eta > 0$ and $\xi > 0$, it follows the assumption $V \in \mathcal{V}_3$ that there is an $L_2(t_0, \eta, \xi) > 0$ such that if $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \xi)}$, then $V(t_0^+, \boldsymbol{x}_0) \le L_2(t_0, \eta, \xi)$. If follows from (4.197) that there is a $\sigma = \sigma(t_0, \eta, \xi) > 0$ such that $\alpha(\sigma) > L_2(t_0, \eta, \xi)$. Choose $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \xi)}$, it follows from (4.198), (4.199) and (4.200) that

$$\begin{aligned} \alpha(d(\boldsymbol{x}(t), \$(t))) &\le V(t, \boldsymbol{x}(t)) \\ &\le V(t_0^+, \boldsymbol{x}_0) \le L_2(t_0, \eta, \xi) < \alpha(\sigma) \end{aligned} \tag{4.205}$$

from which we have $d(\boldsymbol{x}(t), \$(t)) < \sigma$ for $t > t_0$. Therefore, the solutions of system (4.140) are \$-equi-bounded. Therefore, the set \$ is SS1, SB1 and GA1. It follows from Definition 4.5.7 that \$ is a globally equi-asymptotically stable set with respect to system (4.140).

■

Theorem 4.5.10. *Assume that $\alpha, \beta, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w \to \infty} \alpha(w) = \infty, \tag{4.206}$$

$V \in \mathcal{V}_3$, and there are two functions $p_0 : \mathbb{R}_+ \to (0, \infty)$ and $p_1 : \mathbb{R}_+ \to [1, \infty)$ such that

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.207}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.208}$$

$$V(t_0^+, \boldsymbol{x}) \leq p_1(t)\beta(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.209}$$

$$D^+V(t, \boldsymbol{x}) \leq -p_0(t)\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma \tag{4.210}$$

$$\int_0^\infty p_0(s)\kappa[\beta^{-1}(\xi/p_1(s))]ds = \infty$$
$$\text{for any sufficient small } \xi > 0. \tag{4.211}$$

Then \$ is an η-uniformly globally asymptotically stable set with respect to system (4.140). ⊠

Proof. Given $t_0 \in \mathbb{R}_+$ and $\delta > 0$ and let us choose $\varepsilon = \varepsilon(t_0, \delta) \in (0, \delta)$ such that

$$\beta(\varepsilon) < \frac{\alpha(\delta)}{p_1(t_0)}.$$

Choose an arbitrary $\eta > 0$ and let $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \$(t_0, \varepsilon)$, then it follows from (4.207), (4.208), (4.209) and (4.210) that

$$\begin{aligned}\alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \leq V(t_0^+, \boldsymbol{x}_0) \\ &\leq p_1(t_0)\beta(d(\boldsymbol{x}_0, \$(t_0))) \leq p_1(t_0)\beta(\varepsilon) < \alpha(\delta)\end{aligned} \tag{4.212}$$

from which we have $d(\boldsymbol{x}(t), \$(t)) < \delta$ for all $t > t_0$, Therefore $\boldsymbol{x}(t) \in \$(t, \delta)$ for all $t > t_0$; namely, the set \$ is η-uniformly stable.

Given $\xi > 0$, $t_0 \in \mathbb{R}_+$, $\delta > 0$ and let us choose $\phi = \phi(t_0, \xi, \delta) > 0$ such that from (4.211) it follows

$$\int_{t_0}^{t_0+\phi} p_0(s)\kappa\left(\beta^{-1}\left(\frac{\alpha(\delta)}{2p_1(s)}\right)\right) ds > p_1(t_0)\beta(\xi). \tag{4.213}$$

Choose an arbitrary $\eta > 0$ and let $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \xi)}$, we then have the following claim:
Claim 4.5.10: there is a $t_1 \in [t_0, t_0 + \phi]$ such that

$$d(\boldsymbol{x}(t_1), \$(t_1)) < \beta^{-1}\left(\frac{\alpha(\delta)}{2p_1(t_1)}\right).$$

If Claim 4.5.10 is not true then we can assume that for each $t \in [t_0, t_0+\phi]$

$$d(\boldsymbol{x}(t), \$(t)) \geq \beta^{-1}\left(\frac{\alpha(\delta)}{2p_1(t)}\right). \tag{4.214}$$

It follows from (4.210) and (4.213) that

$$\begin{aligned}\int_{t_0}^{t_0+\phi} D^+(s, \boldsymbol{x}(s))ds &\leq -\int_{t_0}^{t_0+\phi} p_0(s)\kappa\left(\beta^{-1}\left(\frac{\alpha(\delta)}{2p_1(s)}\right)\right)ds\\ &< -p_1(t_0)\beta(\xi).\end{aligned} \tag{4.215}$$

Assume that $t+\phi \in (\tau_k, \tau_{k+1}]$ then it follows from (4.208) that

$$\begin{aligned}\int_{t_0}^{t_0+\phi} D^+(s, \boldsymbol{x}(s))ds &= \sum_{i=1}^{k}\int_{\tau_{i-1}^+}^{\tau_i} D^+(s, \boldsymbol{x}(s))ds + \sum_{i=1}^{k}\int_{\tau_k^+}^{t_0+\phi} D^+(s, \boldsymbol{x}(s))ds\\ &= \sum_{i=1}^{k}[V(\tau_i, \boldsymbol{x}(\tau_i)) - V(\tau_{i-1}^+, \boldsymbol{x}(\tau_{i-1}^+))]\\ &\quad +V(t_0+\phi, \boldsymbol{x}(t_0+\phi)) - V(\tau_k^+, \boldsymbol{x}(\tau_k^+))\\ &\geq V(t_0+\phi, \boldsymbol{x}(t_0+\phi)) - V(t_0^+, \boldsymbol{x}_0)\end{aligned} \tag{4.216}$$

It follows from (4.209), (4.215) and (4.216) that

$$V(t_0+\phi, \boldsymbol{x}(t_0+\phi)) < 0$$

which is a contradiction to (4.207). Thus, Claim 4.5.10 is true. Then for $t \geq t_0+\phi$ we have

$$\begin{aligned}\alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \leq V(t_1, \boldsymbol{x}(t_1))\\ &\leq p_1(t_1)\beta(d(t_1, \$(t_1))) \leq \frac{\alpha(\delta)}{2} < \alpha(\delta)\end{aligned} \tag{4.217}$$

form which we have

$$d(\boldsymbol{x}(t), \$(t)) < \delta;$$

namely, $\boldsymbol{x}(t) \in \$(t, \delta)$ for $t \geq t_0+\phi$. This proves that the set \$ is η-uniformly globally attractive with respect to system (4.140).

Given $t_0 \in \mathbb{R}_+$ and $\xi > 0$ and choose $\sigma = \sigma(t_0, \xi) > 0$ such that

$$\alpha(\sigma) > p_1(t_0)\beta(\xi).$$

Let us choose an arbitrary $\eta > 0$, $\boldsymbol{x}_0 \in \mathcal{S}_\eta \cap \overline{\$(t_0, \xi)}$, then for $t > t_0$ we have

$$\begin{aligned}\alpha(d(\boldsymbol{x}(t), \$(t))) &\leq V(t, \boldsymbol{x}(t)) \leq V(t_0, \boldsymbol{x}(t_0))\\ &\leq p_1(t_0)\beta(d(t_0, \$(t_0))) \leq p_1(t_0)\beta(\xi) < \alpha(\sigma)\end{aligned} \tag{4.218}$$

from which it follows that $d(\boldsymbol{x}(t), \$(t)) < \sigma$; namely, $\boldsymbol{x}(t) \in \$(t, \sigma)$ for $t > t_0$. This prove that the solutions of system (4.140) are η-uniformly \$-bounded. Therefore, the set \$ is SS3, SB3 and GA3. It follows from Definition 4.5.7 that \$ is an η-globally equi-asymptotically stable set with respect to system (4.140). ■

Theorem 4.5.11. *Assume that $\alpha, \beta, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w \to \infty} \alpha(w) = \infty, \tag{4.219}$$

$V \in \mathcal{V}_3$, and there is an integrally positive function $p : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.220}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.221}$$

$$V(t_0^+, \boldsymbol{x}) \leq \beta(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.222}$$

$$D^+V(t, \boldsymbol{x}) \leq -p(t)\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma. \tag{4.223}$$

Then \$ is a uniformly globally asymptotically stable set with respect to system (4.140). ⊠

The proof of this theorem can be found in [9]. By using similar proving procedure we can have the following theorems.

Corollary 4.5.3. *Assume that $\alpha, \beta, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w \to \infty} \alpha(w) = \infty, \tag{4.224}$$

$V \in \mathcal{V}_3$ such that

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.225}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.226}$$

$$V(t_0^+, \boldsymbol{x}) \leq \beta(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.227}$$

$$D^+V(t, \boldsymbol{x}) \leq -\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma. \tag{4.228}$$

Then \$ is a uniformly globally asymptotically stable set with respect to system (4.140). ⊠

Corollary 4.5.4. *Assume that $\alpha, \beta, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w \to \infty} \alpha(w) = \infty, \tag{4.229}$$

$V \in \mathcal{V}_3$, and a constant $p > 0$ such that

$$\alpha(d(\boldsymbol{x}, \$(t))) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.230}$$

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.231}$$

$$V(t_0^+, \boldsymbol{x}) \leq \beta(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.232}$$

$$D^+V(t, \boldsymbol{x}) \leq -p\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t, \boldsymbol{x}) \in \Gamma. \tag{4.233}$$

Then \$ is a uniformly globally asymptotically stable set with respect to system (4.140). ⊠

Theorem 4.5.12. *Assume that $\beta(w,\cdot) \in \mathcal{K}$ for each $w \in \mathbb{R}_+$, $\alpha, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w\to\infty} \alpha(w) = \infty, \tag{4.234}$$

$V \in \mathcal{V}_3$, and there is an integrally positive function $p : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}
&\alpha(d(\boldsymbol{x}, \$(t))) \le V(t,\boldsymbol{x}) \text{ for } (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, && (4.235)\\
&V(\tau_k^+, \boldsymbol{x} + U(k,\boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, && (4.236)\\
&V(t_0^+, \boldsymbol{x}) \le \beta(\|\boldsymbol{x}\|, d(\boldsymbol{x}, \$(t))) \text{ for } (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, && (4.237)\\
&D^+V(t,\boldsymbol{x}) \le -p(t)\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t,\boldsymbol{x}) \in \Gamma. && (4.238)
\end{aligned}$$

Then \$ is a t-uniformly globally asymptotically stable set with respect to system (4.140). ⊠

Theorem 4.5.13. *Assume that $\beta(t,\cdot) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$, $\alpha, \kappa \in \mathcal{K}$ satisfying*

$$\lim_{w\to\infty} \alpha(w) = \infty, \tag{4.239}$$

$V \in \mathcal{V}_3$, and there is an integrally positive function $p : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}
&\alpha(d(\boldsymbol{x}, \$(t))) \le V(t,\boldsymbol{x}) \text{ for } (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, && (4.240)\\
&V(\tau_k^+, \boldsymbol{x} + U(k,\boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}) \text{ for } k \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^n, && (4.241)\\
&V(t_0^+, \boldsymbol{x}) \le \beta(t, d(\boldsymbol{x}, \$(t))) \text{ for } (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, && (4.242)\\
&D^+V(t,\boldsymbol{x}) \le -p(t)\kappa(d(\boldsymbol{x}, \$(t))) \text{ for } (t,\boldsymbol{x}) \in \Gamma. && (4.243)
\end{aligned}$$

Then \$ is an η-uniformly globally asymptotically stable set with respect to system (4.140). ⊠

4.6 Stability in Terms of Two Measures

Stability in terms of two measures[11, 16] provides a much more flexible framework of studying the stability of impulsive control systems. For example, to study the stability of a stepping motor or the stability of phase states of a driven single electron tunneling junction, we need to study stability of periodic solutions instead of equilibrium points. In this kind of impulsive control problem, we need the methods presented in this section.

Let us consider the following impulsive control system:

$$\boxed{\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(t,\boldsymbol{x}), \quad t \ne \tau_k(\boldsymbol{x}),\\
\Delta\boldsymbol{x} &= U(k,\boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x})
\end{aligned} \qquad (4.244)}$$

where $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is the nonlinearity of the plant, $\tau_k \in C[\mathbb{R}^n, \mathbb{R}]$ and $U \in C[\mathbb{N} \times \mathbb{R}^n, \mathbb{R}^n]$ is the control impulse.

Definition 4.6.1. *Let $h_0, h \in \mathcal{H}$, then h_0 is* finer *than h if there exist a $\sigma > 0$ and a function $\alpha \in \mathcal{K}$ such that $h_0(t, \boldsymbol{x}) < \sigma$ implies $h(t, \boldsymbol{x}) \leq \alpha(h_0(t, \boldsymbol{x}))$.*

⊠

Definition 4.6.2. *The impulsive system (4.244) is* (h_0, h)-stable *if, given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon)$ such that $h_0(t_0, \boldsymbol{x}_0) < \delta$ implies $h(t, \boldsymbol{x}(t)) < \epsilon$, $t \geq t_0$ for any solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (4.244).*

⊠

Definition 4.6.3. *Let $V \in \mathcal{V}_0$ and $h_0, h \in \mathcal{H}$, then $V(t, \boldsymbol{x})$ is said to be*

1. h-positive definite *if there exist a $\rho > 0$ and a function $\beta \in \mathcal{K}$ such that $h(t, \boldsymbol{x}) < \rho$ implies $\beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x})$;*
2. h_0-decrescent *if there exist a $\delta > 0$ and a function $\alpha \in \mathcal{K}$ such that $h_0(t, \boldsymbol{x}) < \delta$ implies $V(t, \boldsymbol{x}) \leq \alpha(h_0(t, \boldsymbol{x}))$;*
3. weakly h_0-decrescent *if there exist a $\delta > 0$ and a function $\alpha \in \mathcal{CK}$ such that $h_0(t, \boldsymbol{x}) < \delta$ implies $V(t, \boldsymbol{x}) \leq \alpha(t, h_0(t, \boldsymbol{x}))$.*

⊠

Theorem 4.6.1. *Assume that*

1. *$h_0, h \in \mathcal{H}$ and h_0 is finer than h;*
2. *there exists a $V \in \mathcal{V}_0$ such that $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$, and h-positive definite on $\mathcal{S}_\rho(h)$, and $D^+V(t, \boldsymbol{x}) \leq 0$ on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$;*
3. *$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x})$ on $\Sigma_k \cap \mathcal{S}_\rho(h)$.*

Then system (4.244) is

1. *(h_0, h)-stable if $V(t, \boldsymbol{x})$ is weakly h_0-decrescent;*
2. *(h_0, h)-uniformly stable if $V(t, \boldsymbol{x})$ is h_0-decrescent.*

⊠

Proof. If $V(t, \boldsymbol{x})$ is weakly h_0-decrescent, then there exist a $\delta_0 > 0$ and a function $\alpha \in \mathcal{CK}$ such that if $h_0(t, \boldsymbol{x}) < \delta_0$ we have

$$V(t, \boldsymbol{x}) \leq \alpha(t, h_0(t, \boldsymbol{x})). \quad \Leftarrow \text{(Definition 4.6.3)} \tag{4.245}$$

Since $V(t, \boldsymbol{x})$ is h-positive definite on $\mathcal{S}_\rho(h)$, there exists a function $\beta \in \mathcal{K}$ such that

$$\beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in \mathcal{S}_\rho(h). \quad \Leftarrow \text{(Definition 4.6.3)} \tag{4.246}$$

It follows from assumption 1 that there are $\delta_1 > 0$ and $\phi \in \mathcal{K}$ such that if $h_0(t, \boldsymbol{x}) < \delta_1$ we have

$$h(t, \boldsymbol{x}) \leq \phi(h_0(t, \boldsymbol{x})). \quad \Leftarrow \text{(Definition 4.6.1)} \tag{4.247}$$

Given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta_2 \in (0, \delta_0)$ and $\delta_3 \in (0, \delta_1)$ such that

$$\alpha(t_0, \delta_2) < \beta(\epsilon) \text{ and } \phi(\delta_3) < \rho. \tag{4.248}$$

Let us set $\delta = \min(\delta_2, \delta_3)$, choose $\boldsymbol{x}_0$ such that $h_0(t_0, \boldsymbol{x}_0) < \delta$ and let $m(t)$ be $m(t) = V(t, \boldsymbol{x})$. It follows from assumption 2 that $D^+m(t) \leq 0$ for $t \neq \tau_k$. From assumption 3 it follows that $m(\tau_k^+) \leq m(\tau_k)$. Therefore, $m(t)$ is nonincreasing. From (4.245)-(4.248) it follows that for $t \geq t_0$

$$\begin{aligned}
\beta(h(t, \underbrace{\boldsymbol{x}(t))) \leq m}_{(4.246)}(t) \leq m(t_0^+) \qquad &\Leftarrow (m(t) \text{ is nonincreasing}) \\
\leq \alpha(t_0, h_0(t_0, \boldsymbol{x}_0)) \qquad &\Leftarrow (4.245) \\
< \alpha(t_0, \delta) \leq \alpha(t_0, \delta_2) \qquad &\Leftarrow (4.248) \\
< \beta(\epsilon) &
\end{aligned} \tag{4.249}$$

which yields for $t \geq t_0$ and $h_0(t_0, \boldsymbol{x}_0)) < \delta$,

$$h(t, \boldsymbol{x}(t)) < \epsilon. \tag{4.250}$$

We then have conclusion 1.

If $V(t, \boldsymbol{x})$ is h_0-decrescent, we have (4.245) and (4.248) with α independent of t. Thus, δ can be chosen independent of t_0 such that (4.250) holds provided $h_0(t_0, \boldsymbol{x}_0)) < \delta$. This leads to conclusion 2. ■

Theorem 4.6.2. *Assume that*

1. *$h_0, h \in \mathcal{H}$ and h_0 is finer than h;*
2. *there exists a $V \in \mathcal{V}_0$ such that $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$, h-positive definite on $\mathcal{S}_\rho(h)$, weakly h_0-decrescent, and*

$$D^+V(t, \boldsymbol{x}) \leq -\lambda(t)\alpha(V_1(t, \boldsymbol{x}))$$

   on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$, where $\alpha \in \mathcal{K}$, $V_1 \in \mathcal{V}_0$, and $\lambda(t)$ is integrally positive;
3. *$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x}(\tau_k))) \leq V(\tau_k, \boldsymbol{x}(\tau_k))$ on $\Sigma_k \cap \mathcal{S}_\rho(h)$;*
4. *$V_1(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$ and h-positive definite on $\mathcal{S}_\rho(h)$, and for every piecewise continuous function $\boldsymbol{y}(t)$ with discontinuities at $\tau_k = \tau_k(\boldsymbol{y}(\tau_k))$, $k \in \mathbb{N}$, such that $(t, \boldsymbol{y}(t)) \in \mathcal{S}_\rho(h)$, the function*

$$\int_0^t [D^+V_1(s, \boldsymbol{y}(s))]_+ ds \quad (\textit{ resp. } \int_0^t [D^+V_1(s, \boldsymbol{y}(s))]_- ds)$$

   is uniformly continuous on $\mathbb{R}_+$, where $[\cdot]_+$ (resp. $[\cdot]_-$) denotes that the positive (resp. negative) part is considered for all $s \in \mathbb{R}_+$, and

$$V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_1(t, \boldsymbol{x})$$

$$(\textit{resp. } V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq V_1(t, \boldsymbol{x}))$$

   on $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then the impulsive control system (4.244) is (h_0, h)-asymptotically stable. ⊠

Proof. It follows from Theorem 4.6.1 and the assumptions 1-3 that the control system (4.244) is (h_0, h)-stable. Therefore for $\rho > 0$ and $t_0 \in \mathbb{R}_+$, there is a $\delta_0 = \delta_0(t_0, \rho) > 0$ such that $h_0(t_0, \boldsymbol{x}_0) < \delta_0$ implies

$$h(t, \boldsymbol{x}(t)) < \rho, \quad t \geq t_0, \tag{4.251}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is any solution of system (4.244). We then need to prove that for every solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (4.244) satisfying (4.251), the following claim is true:
Claim 4.6.2a:

$$\lim_{t\to\infty} \inf V_1(t, \boldsymbol{x}(t)) = 0. \tag{4.252}$$

If Claim 4.6.2a is not true, then there are a $\xi > 0$ and some $T > 0$ such that

$$V_1(t, \boldsymbol{x}(t)) \geq \xi, \quad t \geq t_0 + T. \tag{4.253}$$

We can choose a sequence

$$t_0 + T < a_1 < b_1 < \cdots < a_i < b_i < \cdots$$

such that $b_i - a_i \geq \xi$ for $i \in \mathbb{N}$. Therefore, from (4.253) and assumption 2 we have

$$\begin{aligned}
\lim_{t\to\infty} V(t, \boldsymbol{x}(t)) &\leq V(t_0, \boldsymbol{x}_0) - \int_{t_0}^{\infty} \lambda(s)\alpha(V_1(s, \boldsymbol{x}(s)))ds \quad \Leftarrow \text{(assumption 2)} \\
&\leq V(t_0, \boldsymbol{x}_0) - \alpha(\xi) \int_{\bigcup_{i=1}^{\infty}[a_i, b_i]} \lambda(s)ds \quad \Leftarrow (4.253) \\
&= -\infty \quad \Leftarrow \text{(assumption 2)}
\end{aligned} \tag{4.254}$$

which leads to a contradiction to the assumption of $V \in \mathcal{V}_0$. Therefore, Claim 4.6.2a is true and (4.252) holds.

We then have the following claim:
Claim 4.6.2b:

$$\lim_{t\to\infty} \sup V_1(t, \boldsymbol{x}(t)) = 0.$$

If Claim 4.6.2b is not true then there is an $\eta > 0$ such that

$$\lim_{t\to\infty} \sup V_1(t, \boldsymbol{x}(t)) > 2\eta.$$

For definiteness, suppose that assumption 4 holds with respect to $[\cdot]_+$. Since (4.252) holds, we can find a sequence

$$t_0 < t_1^{[1]} < t_1^{[2]} < \cdots < t_i^{[1]} < t_i^{[2]} < \cdots$$

such that, for $i \in \mathbb{N}$,

$$V_1(t_i^{[1]}, \boldsymbol{x}(t_i^{[1]})) = \eta, \quad V_1(t_i^{[2]}, \boldsymbol{x}(t_i^{[2]})) = 2\eta, \tag{4.255}$$

from which and since

$$V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_1(\tau_k, \boldsymbol{x}(\tau_k))$$

we can find a sequence $t_0 < a_1 < b_1 < \cdots < a_i < b_i < \cdots$ such that, for $i \in \mathbb{N}$, $t_i^{[1]} \leq a_i < b_i \leq t_i^{[2]}$ and

$$\begin{aligned} &V_1(a_i, \boldsymbol{x}(a_i)) = \eta, \quad V_1(b_i, \boldsymbol{x}(b_i)) = 2\eta, \\ &V_1(t, \boldsymbol{x}(t)) \in [\eta, 2\eta] \text{ for } t \in [a_i, b_i]. \end{aligned} \tag{4.256}$$

We then have

$$\begin{aligned} 0 < \eta &= V_1(b_i, \boldsymbol{x}(b_i)) - V_1(a_i, \boldsymbol{x}(a_i)) \\ &\leq \int_{a_i}^{b_i} [D^+ V_1(s, \boldsymbol{x}(s))]_+ ds, \qquad i \in \mathbb{N} \end{aligned} \tag{4.257}$$

from which and assumption 4, we have for some constant $c > 0$

$$b_i - a_i \geq c, \qquad i \in \mathbb{N}. \tag{4.258}$$

Therefore, from assumption 2, (4.256) and (4.258) we have

$$\begin{aligned} \lim_{t \to \infty} V(t, \boldsymbol{x}(t)) &\leq V(t_0, \boldsymbol{x}_0) - \int_{t_0}^{\infty} \lambda(s) \alpha(V_1(s, \boldsymbol{x}(s))) ds \qquad \Leftarrow \text{(assumption 2)} \\ &\leq V(t_0, \boldsymbol{x}_0) - \alpha(\eta) \int_{\bigcup_{i=1}^{\infty} [a_i, b_i]} \lambda(s) ds \qquad \Leftarrow (4.256) \\ &= -\infty. \qquad \Leftarrow (4.258) \end{aligned} \tag{4.259}$$

This is a contradiction to $V \in \mathcal{V}_0$. Therefore, Claim 4.6.2b is true. It follows from Claim 4.6.2a and Claim 4.6.2b that

$$\lim_{t \to \infty} V_1(t, \boldsymbol{x}(t)) = 0, \tag{4.260}$$

and since $V_1(t, \boldsymbol{x}(t))$ is h-positive definite, it follows from Definition 4.6.3 that there is a $\beta \in \mathcal{K}$ such that

$$\lim_{t \to \infty} \beta(h(t, \boldsymbol{x}(t))) \leq \lim_{t \to \infty} V_1(t, \boldsymbol{x}(t)) = 0$$

from which we have

$$\lim_{t \to \infty} h(t, \boldsymbol{x}(t)) = 0.$$

This proves that the impulsive control system (4.244) is (h_0, h)-asymptotically stable. ■

From Theorem 4.6.2 it follows the following corollaries.

Corollary 4.6.1. *Assume that*

1. *$h_0, h \in \mathcal{H}$ and h_0 is finer than h;*
2. *there exists a $V \in \mathcal{V}_0$ such that $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$, h-positive definite on $\mathcal{S}_\rho(h)$, weakly h_0-decrescent, and*
$$D^+V(t, \boldsymbol{x}) \leq -\lambda(t)\alpha(V_1(t, \boldsymbol{x}))$$
on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$, where $\alpha \in \mathcal{K}$, $V_1 \in \mathcal{V}_0$, and $\lambda(t)$ is integrally positive;
3. *$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x}(\tau_k)) \leq V(\tau_k, \boldsymbol{x}(\tau_k))$ on $\Sigma_k \cap \mathcal{S}_\rho(h)$;*
4. *$V_1(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$ and h-positive definite on $\mathcal{S}_\rho(h)$. $D^+V_1(t, \boldsymbol{x})$ is bounded from above(resp. from below) on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$ and*
$$V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_1(\tau_k, \boldsymbol{x}(\tau_k))$$
$$(resp.\ V_1(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq V_1(\tau_k, \boldsymbol{x}(\tau_k))\ )$$
on $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then system (4.244) is (h_0, h)-asymptotically stable. ⊠

Corollary 4.6.2. *Assume that*

1. *$h_0, h \in \mathcal{H}$ and h_0 is finer than h;*
2. *there exists a $V \in \mathcal{V}_0$ such that $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$, h-positive definite on $\mathcal{S}_\rho(h)$, weakly h_0-decrescent, and*
$$D^+V(t, \boldsymbol{x}) \leq -\lambda(t)\alpha(V(t, \boldsymbol{x}))$$
on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$, where $\alpha \in \mathcal{K}$ and $\lambda(t)$ is integrally positive;
3. *$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x}(\tau_k))) \leq V(\tau_k, \boldsymbol{x}(\tau_k))$ on $\Sigma_k \cap \mathcal{S}_\rho(h)$.*

Then system (4.244) is (h_0, h)-asymptotically stable. ⊠

Proof. Let us set $V_1(t, \boldsymbol{x})$ in Theorem 4.6.2 as $V(t, \boldsymbol{x})$, then conditions 1-3 of Theorem 4.6.2 are satisfied. Furthermore, it follows from conditions 2 and 3 we immediately have condition 4 of Theorem 4.6.2. This finishes the proof. ■

Theorem 4.6.3. *Assume that*

1. *$h_0, h \in \mathcal{H}$ and h_0 is finer than h;*
2. *there exists a $V \in \mathcal{V}_0$ such that $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ on each $\mathfrak{G}_i$, h-positive definite on $\mathcal{S}_\rho(h)$, h_0-decrescent, and*
$$D^+V(t, \boldsymbol{x}) \leq -\lambda(t)\alpha(h_0(t, \boldsymbol{x}))$$
on $\mathfrak{G} \cap \mathcal{S}_\rho(h)$, where $\alpha \in \mathcal{K}$ and $\lambda(t)$ is integrally positive;
3. *$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x}(\tau_k))) \leq V(\tau_k, \boldsymbol{x}(\tau_k))$ on $\Sigma_k \cap \mathcal{S}_\rho(h)$.*

Then system (4.244) is (h_0, h)-uniformly asymptotically stable. ⊠

Proof. From Theorem 4.6.1 it follows that the control system (4.244) is (h_0, h)-uniformly stable. Therefore for $\rho > 0$ there is a $\delta_0 = \delta_0(\rho) > 0$ such that $h_0(t_0, \boldsymbol{x}_0) < \delta_0$ implies

$$h(t, \boldsymbol{x}(t)) < \rho, \qquad t \geq t_0, \tag{4.261}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is any solution of system (4.244).

Given $0 < \eta < \rho$, let $\delta = \delta(\eta) > 0$, then we have the following claim:
Claim 4.6.3: There is a $T = T(\eta) > 0$ such that, for some $t^* \in [t_0, t_0 + T]$, we have

$$h_0(t^*, \boldsymbol{x}(t^*)) < \delta. \tag{4.262}$$

If Claim 4.6.3 is not true, then for any $T > 0$ there is a solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (4.244) satisfying (4.261) such that

$$h_0(t, \boldsymbol{x}(t)) \geq \delta, \qquad t \in [t_0, t_0 + T]. \tag{4.263}$$

Since $V(t, \boldsymbol{x}(t))$ is nonincreasing, from assumption 2 we know that there is a $\beta \in \mathcal{K}$ such that

$$\begin{aligned}
\int_{t_0}^{\infty} \lambda(s)\alpha(h_0(s, \boldsymbol{x}(s)))ds &\leq -\int_{t_0}^{\infty} D^+V(s, \boldsymbol{x}(s))ds \qquad \Leftarrow \text{(assumption 2)} \\
&= V(t_0, \boldsymbol{x}_0) - V(\infty, \boldsymbol{x}(\infty)) \\
&= V(t_0, \boldsymbol{x}_0) \qquad \Leftarrow [\text{procedure leads to (4.260)}] \\
&\leq \beta(\delta_0) \qquad \Leftarrow (V \text{ is } h_0\text{-decrescent})
\end{aligned} \tag{4.264}$$

for each solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (4.244) satisfying (4.261). From the assumption on $\lambda(t)$ it follows that there is a $T > 0$ such that

$$\int_{t_0}^{t_0+T} \lambda(s)ds > \frac{\beta(\delta_0) + 1}{\alpha(\delta)}. \tag{4.265}$$

Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be the solution of system (4.244) satisfying (4.263) with T given in (4.265), then from (4.264) and (4.265) we have

$$\begin{aligned}
\beta(\delta_0) &\geq \int_{t_0}^{\infty} \lambda(s)\alpha(h_0(s, \boldsymbol{x}(s))ds \qquad \Leftarrow (4.264) \\
&> \alpha(\delta)\int_{t_0}^{t_0+T} \lambda(s)ds \qquad \Leftarrow (4.263) \\
&> \beta(\delta_0) + 1 \qquad \Leftarrow (4.265)
\end{aligned}$$

which is a contradiction. Therefore, Claim 4.6.3 is true. This proves that the impulsive control system (4.244) is (h_0, h)-uniformly asymptotically stable. ■

Note 4.6.1. The results of Lyapunov second method are adopted from [2, 27]. An earlier result based on Lyapunov second method can be found in [4]. Some recent results can be found in [8]. Stability in terms of two measures can be found in [16, 11, 13]. ♦

5. Impulsive Control with Impulses at Variable Time

In this chapter we study impulsive control systems with impulses at variable time. In this kind of control problem, the impulses are generated based on conditions that depend on either plants or control laws. Therefore, the moments of impulses are not necessarily the same for different solutions.

5.1 Linear Decomposition Methods

Consider the following impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}), \end{aligned} \qquad (5.1)}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is bounded and continuous for $t \geq t_0$. $\boldsymbol{g}(t, \boldsymbol{x})$ is continuous with respect to $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\| \leq h, h > 0$. $\boldsymbol{g}(t, \boldsymbol{x})$ is continuous or piecewise continuous with respect to t, $t \geq t_0$. $\boldsymbol{u}_k(\boldsymbol{x}), k \in \mathbb{N}$ is continuous for $\|\boldsymbol{x}\| \leq h$. Observe that the moments of control impulses are different for different solutions. The corresponding reference system is given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x}, \quad t \neq \tau_k^*, \\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x}, \quad t = \tau_k^*, \end{aligned} \qquad (5.2)$$

where τ_k^* are such that for $\|\boldsymbol{x}\| \leq h$, $h > 0$ we have

$$|\tau_k^* - \tau_k(0)| \leq \eta \qquad (5.3)$$

where $\eta = \eta(h) > 0$ satisfies

$$\lim_{h\to 0} \eta(h) = 0. \qquad (5.4)$$

Theorem 5.1.1. *Assume that*

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 119−147, 2001.

1. for all $\|\boldsymbol{x}\| \le h$, $i \in \mathbb{N}$ *and some* $\theta > 0$

$$\sup_i \left(\min_{\|\boldsymbol{x}\| \le h} \tau_{i+1}(\boldsymbol{x}) - \max_{\|\boldsymbol{x}\| \le h} \tau_i(\boldsymbol{x}) \right) \ge \theta; \tag{5.5}$$

2. for all $\|\boldsymbol{x}\| \le h$ *and* $i \in \mathbb{N}$

$$\tau_i(\boldsymbol{x}) \ge \tau_i((I + B_i)\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.6}$$

3. for all $t \ge t_0$, $i \in \mathbb{N}$ *and* $\|\boldsymbol{x}\| \le h$ *we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \le a\|\boldsymbol{x}\|, \qquad \|\boldsymbol{u}_i(\boldsymbol{x})\| \le a\|\boldsymbol{x}\|, \qquad a > 0; \tag{5.7}$$

4. the state transition matrix $\Psi(t, s)$ *of reference system (5.2) satisfies*

$$\|\Psi(t, s)\| \le K e^{-\gamma(t-s)}, \quad t \ge s, \quad K \ge 1, \quad \gamma > 0; \tag{5.8}$$

5.

$$\gamma - Ka - \frac{1}{\theta} \ln(1 + Ka) > 0. \tag{5.9}$$

Then the zero solution of system (5.1) is asymptotically stable. ⊠

Proof. Let $\boldsymbol{x}(t)$ be an arbitrary solution of system (5.1) that at $t = t_0$ passes through a point $\boldsymbol{x}_0$ which is in a small neighborhood of $\boldsymbol{x} = 0$. There exist $\tilde{h} < h$ and $T \le \infty$ such that $\|\boldsymbol{x}(t)\| < h$ for $t \in (t_0, t_0 + T]$ and $\|\boldsymbol{x}_0\| \le \tilde{h}$. Let τ_i^* be a solution of the equation $t = \tau_i(\boldsymbol{x}(t))$ which has a unique solution for all i if assumption 2 holds. It follows from assumption 1 that $\boldsymbol{x}(t)$ intersects each surface $t = \tau_i(\boldsymbol{x})$ only once for $t \in (t_0, t_0 + T]$. Then for $t \in (t_0, t_0 + T]$, $\boldsymbol{x}(t)$ is also a solution of the following system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \qquad t \ne \tau_i^*, \\ \Delta \boldsymbol{x} &= B_i \boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x}), \qquad t = \tau_i^* \end{aligned} \tag{5.10}$$

Then $\boldsymbol{x}(t), t \in (t_0, t_0 + T]$ can be represented as

$$\begin{aligned} \boldsymbol{x}(t) = \Psi(t, t_0)\boldsymbol{x}_0 &+ \int_{t_0}^{t} \Psi(t, s)\boldsymbol{g}(s, \boldsymbol{x}(s))ds \\ &+ \sum_{t_0 < \tau_i^* < t} \Psi(t, \tau_i^*)\boldsymbol{u}_i(\boldsymbol{x}(\tau_i^*)), \end{aligned} \tag{5.11}$$

It follows from assumptions 3 and 4 that

$$\begin{aligned} \|\boldsymbol{x}(t)\| \le K e^{-\gamma(t-t_0)}\|\boldsymbol{x}_0\| &+ Ka \int_{t_0}^{t} e^{-\gamma(t-s)}\|\boldsymbol{x}(s)\|ds \\ &+ \sum_{t_0 < \tau_i^* < t} Ka e^{-\gamma(t-\tau_i^*)}\|\boldsymbol{x}(\tau_i^*)\| \end{aligned} \tag{5.12}$$

from which and Lemma 1.7.1 we have

$$\|\boldsymbol{x}(t)\| \leq K\|\boldsymbol{x}_0\|e^{-(\gamma-Ka)(t-t_0)}(1+Ka)^{\mathfrak{N}(t_0,t)}. \tag{5.13}$$

From assumption 1 it follows that the surfaces $t = \tau_i(\boldsymbol{x}), i \in \mathbb{N}$, are mutually separated and from (5.13) we have

$$\|\boldsymbol{x}(t)\| \leq K\|\boldsymbol{x}_0\| \exp\left\{-\left(\gamma - Ka - \frac{1}{\theta}\ln(1+Ka)\right)(t-t_0)\right\}. \tag{5.14}$$

If $\boldsymbol{x}_0$ satisfies $K\|\boldsymbol{x}_0\| < h$ and assumption 5 holds, then $\boldsymbol{x}(t)$ will not leave the h-neighborhood of $\boldsymbol{x} = 0$ for all $t \geq t_0$. Therefore it follows from assumptions 1 and 2 that $\boldsymbol{x}(t)$ intersects each surface $t = \tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, only once. Then from (5.14) and assumption 5 it follows that the zero solution of system (5.1) is asymptotically stable. ■

Similarly we have the following theorem.

Theorem 5.1.2. *Assume that*

1. *the largest eigenvalue of* $\frac{1}{2}(A(t) + A^\top(t))$, $\lambda_n(t)$, *satisfies* $\lambda_n(t) \leq \gamma$ *for all* $t \geq t_0$ *and the largest eigenvalues of* $(I + B_i^\top)(I + B_i)$, Λ_i, $i \in \mathbb{N}$, *satisfy* $\Lambda_i^2 \leq \alpha^2$;
2. $n_x(t, t+T)$ *is the number of points of* $\tau_i(\boldsymbol{x})$ *in time interval* $[t, t+T]$ *and for* $\|\boldsymbol{x}\| \leq h$ *the limit*

$$\lim_{T\to\infty} \frac{n_x(t,t+T)}{T} = p$$

   exists and is uniform with respect to $t \geq t_0$;
3. *functions* $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, *satisfy Lipschitz condition*

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \leq L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \qquad 0 < L < \infty \tag{5.15}$$

   for all $\|\boldsymbol{x}_1\| \leq h$ *and* $\|\boldsymbol{x}_2\| \leq h$;
4. *for all* $\|\boldsymbol{x}\| \leq h$ *and* $i \in \mathbb{N}$

$$\tau_i(\boldsymbol{x}) \geq \tau_i((I + B_i)\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.16}$$

5. $\gamma + p\ln\alpha < 0$;
6. *for all* $t \geq t_0$ *and* $i \in \mathbb{N}$ *and* $\|\boldsymbol{x}\| \leq h$ *we have*

$$\|\boldsymbol{g}(t,\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|\boldsymbol{u}_i(\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{5.17}$$

Then the zero solution of system (5.1) is asymptotically stable for sufficiently small values of a. ⊠

Proof. Let $\boldsymbol{x}(t)$ be an arbitrary solution of system (5.1) that at $t = t_0$ passes through a point $\boldsymbol{x}_0$ which is in a small neighborhood of $\boldsymbol{x} = 0$. There exist $\tilde{h} < h$ and $T \leq \infty$ such that $\|\boldsymbol{x}(t)\| < h$ for $t \in (t_0, t_0 + T]$ and $\|\boldsymbol{x}_0\| \leq \tilde{h}$.

Let τ_i^* be a solution of the equation $t = \tau_i(\boldsymbol{x}(t))$ which has a unique solution for all i if assumptions 3 and 4 hold. This means that $\boldsymbol{x}(t)$ intersects each surface $t = \tau_i(\boldsymbol{x})$ only once for $t \in (t_0, t_0 + T]$. Then for $t \in (t_0, t_0 + T]$, $\boldsymbol{x}(t)$ is also a solution of the following system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \qquad t \neq \tau_i^*, \\ \Delta \boldsymbol{x} &= B_i \boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x}), \qquad t = \tau_i^* \end{aligned} \tag{5.18}$$

Thus $\boldsymbol{x}(t), t \in (t_0, t_0 + T]$ can be represented as

$$\begin{aligned} \boldsymbol{x}(t) = \Psi(t, t_0)\boldsymbol{x}_0 &+ \int_{t_0}^{t} \Psi(t, s)\boldsymbol{g}(s, \boldsymbol{x}(s))ds \\ &+ \sum_{t_0 < \tau_i^* < t} \Psi(t, \tau_i^*)\boldsymbol{u}_i(\boldsymbol{x}(\tau_i^*)), \end{aligned} \tag{5.19}$$

Using the similar proving process of Theorem 4.2.2 and based on assumptions 1, 2 and 5 we know that there exist $K \geq 1$, $\mu > 0$ with $0 < \mu < |\gamma + p \ln \alpha|$ such that for all $t_0 \leq s \leq t \leq t + T$

$$\|\Psi(t, s)\boldsymbol{z}\| \leq K e^{-\mu(t-s)} \|\boldsymbol{z}\| \text{ provided } \|\boldsymbol{z}\| \leq h. \tag{5.20}$$

From (5.19), (5.20), assumption 6 and Lemma 1.7.1 it follows that

$$\|\boldsymbol{x}(t)\| \leq e^{-\mu(t-t_0)} K \|\boldsymbol{x}_0\| (1 + Ka)^{\mathfrak{N}(t_0, t)} e^{Ka(t-t_0)}, \tag{5.21}$$

from which we have

$$\|\boldsymbol{x}(t)\| \leq K_1 \|\boldsymbol{x}_0\| \exp\{-[\mu - p \ln(1 + Ka) - Ka + \epsilon](t - t_0)\} \tag{5.22}$$

for any $\epsilon > 0$ with $K_1 = K_1(\epsilon) > 0$. Therefore, if a is sufficiently small such that

$$\mu - p \ln(1 + Ka) - Ka > 0$$

and if $\boldsymbol{x}_0$ satisfies $K_1 \|\boldsymbol{x}_0\| < h$, then $\boldsymbol{x}(t)$ will not leave the h-neighborhood of $\boldsymbol{x} = 0$ for all $t \geq t_0$. Therefore in follows from assumptions 3 and 4 that $\boldsymbol{x}(t)$ intersects each surface $t = \tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, only once. Then from (5.22) it follows that the zero solution of system (5.1) is asymptotically stable if a is sufficiently small. ■

In parallel we have

Theorem 5.1.3. *Assume that*

1. *the largest eigenvalue of $\frac{1}{2}(A(t) + A^\top(t))$, $\lambda_n(t)$, satisfies $\lambda_n(t) \leq \gamma$ for all $t \geq t_0$ and the largest eigenvalues of $(I + B_i^\top)(I + B_i)$, Λ_i, $i \in \mathbb{N}$, satisfy $\Lambda_i^2 \leq \alpha^2$;*
2. $$0 < \theta_1 \leq \min_{\|\boldsymbol{x}\| \leq h} \tau_{i+1}(\boldsymbol{x}) - \max_{\|\boldsymbol{x}\| \leq h} \tau_i(\boldsymbol{x}) \leq \theta_2, \qquad i \in \mathbb{N};$$

3. *$\gamma + \frac{1}{\theta}\ln\alpha < 0$, $\theta = \theta_1$ if $\alpha \geq 1$, and $\theta = \theta_2$ if $0 < \alpha < 1$;*
4. *functions $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, satisfy Lipschitz condition*

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \leq L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad 0 < L < \infty \tag{5.23}$$

   for all $\|\boldsymbol{x}_1\| \leq h$ and $\|\boldsymbol{x}_2\| \leq h$;
5. *for all $\|\boldsymbol{x}\| \leq h$ and $i \in \mathbb{N}$*

$$\tau_i(\boldsymbol{x}) \geq \tau_i((I + B_i)\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.24}$$

6. *for all $t \geq t_0$, $i \in \mathbb{N}$ and $\|\boldsymbol{x}\| \leq h$ we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \quad \|\boldsymbol{u}_i(\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{5.25}$$

Then the zero solution of system (5.1) is asymptotically stable for sufficiently small values of a. ⊠

If $A(t)$ and B_i are time-invariant, then system (5.1) becomes

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \quad t \neq \tau_k(\boldsymbol{x}), \\ \Delta\boldsymbol{x} &= B\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}), \quad t = \tau_k(\boldsymbol{x}) \end{aligned} \qquad (5.26)}$$

whose stability can be guaranteed by the following theorem.

Theorem 5.1.4. *Assume that*

1. *$n_x(t, t+T)$ is the number of points of $\tau_i(\boldsymbol{x})$ in time interval $[t, t+T]$ and for $\|\boldsymbol{x}\| \leq h$ the limit*

$$\lim_{T\to\infty} \frac{n_x(t, t+T)}{T} = p$$

   exists and is uniform with respect to $t \geq t_0$;
2.

$$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A), \quad \alpha^2 = \max_{j=1}^{n} \mathrm{Re}\lambda_j[(I + B^\top)(I + B)];$$

3. *$\gamma + p\ln\alpha < 0$;*
4. *for all $\|\boldsymbol{x}\| \leq h$ and $i \in \mathbb{N}$*

$$\tau_i(\boldsymbol{x}) \geq \tau_i((I + B)\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.27}$$

5. *functions $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, satisfy Lipschitz condition*

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \leq L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad 0 < L < \infty \tag{5.28}$$

   for all $\|\boldsymbol{x}_1\| \leq h$ and $\|\boldsymbol{x}_2\| \leq h$;

6. *for all* $t \geq t_0$ *and* $i \in \mathbb{N}$ *and* $\|\boldsymbol{x}\| \leq h$ *we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|\boldsymbol{u}_i(\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{5.29}$$

Then the zero solution of system (5.26) is asymptotically stable for sufficiently small values of a. ⊠

In parallel we have

Theorem 5.1.5. *Assume that*

1. $$\gamma = \max_{j=1}^{n} \mathrm{Re}\lambda_j(A), \qquad \alpha^2 = \max_{j=1}^{n} \mathrm{Re}\lambda_j[(I + B^\top)(I + B)];$$
2. $$0 < \theta_1 \leq \min_{\|\boldsymbol{x}\| \leq h} \tau_{i+1}(\boldsymbol{x}) - \max_{\|\boldsymbol{x}\| \leq h} \tau_i(\boldsymbol{x}) \leq \theta_2, \qquad i \in \mathbb{N};$$
3. $\gamma + \frac{1}{\theta} \ln \alpha < 0$, $\theta = \theta_1$ *if* $\alpha \geq 1$, *and* $\theta = \theta_2$ *if* $0 < \alpha < 1$;
4. *for all* $\|\boldsymbol{x}\| \leq h$ *and* $i \in \mathbb{N}$

$$\tau_i(\boldsymbol{x}) \geq \tau_i((I + B)\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.30}$$

5. *functions* $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, *satisfy Lipschitz condition*

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \leq L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \qquad 0 < L < \infty \tag{5.31}$$

   for all $\|\boldsymbol{x}_1\| \leq h$ *and* $\|\boldsymbol{x}_2\| \leq h$;
6. *for all* $t \geq t_0$, $i \in \mathbb{N}$ *and* $\|\boldsymbol{x}\| \leq h$ *we have*

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \leq a\|\boldsymbol{x}\|, \qquad \|\boldsymbol{u}_i(\boldsymbol{x})\| \leq a\|\boldsymbol{x}\|. \tag{5.32}$$

Then the zero solution of system (5.26) is asymptotically stable for sufficiently small values of a. ⊠

Furthermore, if the system (5.26) is simplified as

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}), \qquad t \neq \tau_k(\boldsymbol{x}), \\ \Delta\boldsymbol{x} &= \boldsymbol{u}_k(\boldsymbol{x}), \qquad t = \tau_k(\boldsymbol{x}) \end{aligned} \qquad (5.33)}$$

we then have the following theorems.

Theorem 5.1.6. *Assume that*

1. $n_x(t, t+T)$ *is the number of points of* $\tau_i(\boldsymbol{x})$ *in time interval* $[t, t+T]$ *and for* $\|\boldsymbol{x}\| \leq h$ *the limit*

$$\lim_{T \to \infty} \frac{n_x(t, t+T)}{T} = p$$

   exists and is uniform with respect to $t \geq t_0$;

2. all eigenvalues of A are in the left half s-plane;
3. for all $\|\boldsymbol{x}\| \le h$ and $i \in \mathbb{N}$

$$\tau_i(\boldsymbol{x}) \ge \tau_i(\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.34}$$

4. functions $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, satisfy Lipschitz condition

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \le L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad 0 < L < \infty, \tag{5.35}$$

for all $\|\boldsymbol{x}_1\| \le h$, $\|\boldsymbol{x}_2\| \le h$;
5. for all $t \ge t_0$, $i \in \mathbb{N}$ and $\|\boldsymbol{x}\| \le h$ we have

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \le a\|\boldsymbol{x}\|, \qquad \|\boldsymbol{u}_i(\boldsymbol{x})\| \le a\|\boldsymbol{x}\|. \tag{5.36}$$

Then the zero solution of system (5.33) is asymptotically stable for sufficiently small values of a. ⊠

In parallel we have

Theorem 5.1.7. *Assume that*

1. for $i \in \mathbb{N}$

$$0 < \theta_1 \le \min_{\|\boldsymbol{x}\| \le h} \tau_{i+1}(\boldsymbol{x}) - \max_{\|\boldsymbol{x}\| \le h} \tau_i(\boldsymbol{x}) \le \theta_2; \tag{5.37}$$

2. all eigenvalues of A are in left half s-plane;
3. for all $\|\boldsymbol{x}\| \le h$ and $i \in \mathbb{N}$

$$\tau_i(\boldsymbol{x}) \ge \tau_i(\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})); \tag{5.38}$$

4. functions $\tau_i(\boldsymbol{x})$, $i \in \mathbb{N}$, satisfy Lipschitz condition

$$\|\tau_i(\boldsymbol{x}_1) - \tau_i(\boldsymbol{x}_2)\| \le L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \qquad 0 < L < \infty \tag{5.39}$$

for all $\|\boldsymbol{x}_1\| \le h$ and $\|\boldsymbol{x}_2\| \le h$;
5. for all $t \ge t_0$, $i \in \mathbb{N}$ and $\|\boldsymbol{x}\| \le h$ we have

$$\|\boldsymbol{g}(t, \boldsymbol{x})\| \le a\|\boldsymbol{x}\|, \quad \|\boldsymbol{u}_i(\boldsymbol{x})\| \le a\|\boldsymbol{x}\|. \tag{5.40}$$

Then the zero solution of system (5.33) is asymptotically stable for sufficiently small values of a. ⊠

5.2 Methods Based on Two Measures

Let us consider the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}), & t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & k = 1, 2, \cdots . \end{cases} \tag{5.41}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is the nonlinearity of the uncontrolled plant, $\boldsymbol{u} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is the continuous control input, $U(k, \boldsymbol{x}) \in C[\mathbb{N} \times \mathbb{R}^n, \mathbb{R}^n]$ is the impulsive control input, $\tau_k \in C^1[\mathbb{R}^n, (0, \infty)]$, $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ for all k and $\lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty$ for every $\boldsymbol{x} \in \mathbb{R}^n$. Let us assume that the solutions $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (5.41) exist and are unique for $t \geq t_0$, and each solution of (5.41) hits any given switching surface $\Sigma_k : t = \tau_k(\boldsymbol{x})$ exactly once.

For studying the stability of impulsive control systems with control impulses at variable time, we need to consider the following comparison system:

$$\begin{aligned} \dot{w} &= g(t, w), \quad t \notin [\tau_k^a, \tau_k^b], \quad k \in \mathbb{N}, \\ w(\tau_k^{b+}) &= \psi_k(w(\tau_k^a)), \\ w(t_0) &= w_0 \end{aligned} \tag{5.42}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $\psi_k \in C[\mathbb{R}, \mathbb{R}]$ and

$$0 \leq t_0 < \tau_1^a \leq \tau_1^b < \tau_2^a \leq \tau_2^b < \cdots < \tau_k^a \leq \tau_k^b < \cdots, \quad \lim_{k\to\infty} \tau_k^a = \infty.$$

A general form of the solutions of (5.42) is given by

$$w(t, t_0, w_0) = \begin{cases} w_0(t, t_0, w_0), & t \in [t_0, \tau_1^a], \\ w_1(t, \tau_1^b, w_1^+), & t \in (\tau_1^b, \tau_2^a], \\ w_2(t, \tau_2^b, w_2^+), & t \in (\tau_2^b, \tau_3^a], \\ \vdots & \vdots \\ w_k(t, \tau_k^b, w_k^+), & t \in (\tau_k^b, \tau_{k+1}^a], \\ w_{k+1}(t, \tau_{k+1}^b, w_{k+1}^+), & t \in (\tau_{k+1}^b, \tau_{k+2}^a], \\ \vdots & \vdots \end{cases} \tag{5.43}$$

where $w_0^+ = w_0$ and $w_{k+1}(t, \tau_{k+1}^b, w_{k+1}^+)$ is a solution of

$$\dot{w} = g(t, w) \tag{5.44}$$

with

$$w_{k+1}^+ = \psi_{k+1}(w_k(\tau_{k+1}^a, \tau_k^b, w_k^+)), \quad t \in [\tau_k^b, \tau_{k+1}^a].$$

Theorem 5.2.1. *Let us suppose that the following conditions hold:*

1. *$w \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ with points of discontinuity at $t = \tau_k^a$, $t = \tau_k^b$, $w(t)$ is left continuous at $t = \tau_k^a$ and*

$$\begin{aligned} D^+ w(t) &\leq g(t, w(t)), \quad t \notin [\tau_k^a, \tau_k^b], \quad k \in \mathbb{N}, \\ w(\tau_k^{b+}) &\leq \psi_k(w(\tau_k^a)), \\ w(t_0) &\leq w_0 \end{aligned} \tag{5.45}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $\psi_k \in C[\mathbb{R}, \mathbb{R}]$ and $\psi_k(w)$ is nondecreasing in w;

2. *$w_{\max}(t, t_0, w_0)$ is the maximal solution of (5.42) on*

$$[t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\tau_k^a, \tau_k^b].$$

Then

$$w(t) \leq w_{\max}(t, t_0, w_0) \textit{ for } t \in [t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\tau_k^a, \tau_k^b]. \tag{5.46}$$

$\boxtimes$

Proof. We use mathematical induction to prove this theorem. Let $w_{\max}(t, t_0, w_0)$ be the maximal solution of (5.42), then from the classical comparison theorem we have for $w(t_0) \leq w_0$

$$w(t) \leq w_{\max 0}(t, t_0, w_0), \quad t \in [t_0, \tau_1^a]$$

where $w_{\max 0}(t, t_0, w_0)$ is the maximal solution of (5.44) on $t \in [t_0, \tau_1^a]$ with $w_{\max 0}(t_0, t_0, w_0) = w_0$.

Then let us assume that

$$w(t) \leq w_{\max i}(t, \tau_i^b, w_i^+), \quad t \in [\tau_i^b, \tau_{i+1}^a]$$

where $w_{\max i}(t, \tau_i^b, w_i^+)$ is the maximal solution of (5.44) on $t \in [\tau_i^b, \tau_{i+1}^a]$ such that $w(\tau_i^{b+}) \leq w_i^+$ for some $i \geq 1$. Then we have

$$w(\tau_{i+1}^a) \leq w_{\max i}(\tau_{i+1}^a, \tau_i^b, w_i^+)$$

and since $\psi_{i+1}(w)$ is nonincreasing in w, we have

$$w(\tau_{i+1}^b) \leq \psi_{i+1}(w_{\max i}(\tau_{i+1}^a, \tau_i^b, w_i^+)) = w_{i+1}^+.$$

Then it follows classical comparison theorem that

$$w(t) \leq w_{\max(i+1)}(t, \tau_{i+1}^b, w_{i+1}^+), \quad t \in [\tau_{i+1}^b, \tau_{i+2}^a]$$

where $w_{\max(i+1)}(t, \tau_{i+1}^b, w_{i+1}^+)$ is the maximal solution of (5.44) with $w_{\tau_{i+1}^{b+}} \leq w_{i+1}^+$ for $t \in [\tau_{i+1}^b, \tau_{i+2}^a]$.

Therefore, by using mathematical induction, we have

$$w(t) \leq w_{\max k}(t, \tau_k^b, w_k^+), \quad t \in [\tau_k^b, \tau_{k+1}^a], \quad k \in \mathbb{N}.$$

This completes the proof. $\blacksquare$

Theorem 5.2.2. *Given any a solution,* $\boldsymbol{x}_a(t)$, $t \in [t_0, \infty)$, *of (5.41) with* $\boldsymbol{x}_a(t_0^+) = \boldsymbol{x}_{a0}$ *and assume* $\boldsymbol{x}_a(t)$ *hits the switching surface* Σ_i *at moments* τ_i^a, $i \in \mathbb{N}$. *Let* $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ *be any solution of (5.41) and it hits the switching surface* Σ_i *at* $t = \tau_i$, $i \in \mathbb{N}$. *We assume that*

1. $\|\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x})\| \le L_1$ *on* $\mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$;
2. *for all* $k \in \mathbb{N}$ *and for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$,

$$\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}}[\boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x})] \le 0$$

and

$$|\tau_k(\boldsymbol{x}) - \tau_k(\boldsymbol{y})| \le K_k\|\boldsymbol{x} - \boldsymbol{y}\|, K_k \ge 0;$$

3. $V(t,\boldsymbol{x}) \in C[\mathbb{R}_+ \times \mathbb{R}_n, \mathbb{R}_+]$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$, *we have*

$$\begin{aligned} &D^+V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)) \le g(t, V(t, \boldsymbol{x} - \boldsymbol{x}_a(t))), \\ &t \ne \tau_k^a, \quad t \ne \tau_k(\boldsymbol{x}), \end{aligned} \tag{5.47}$$

$$\begin{aligned} &V(t, \boldsymbol{x} - \boldsymbol{y} + U(k, \boldsymbol{x}) - U(k, \boldsymbol{y})) \le \nu_k(V(t, \boldsymbol{x} - \boldsymbol{y})), \\ &t \ne \tau_k(\boldsymbol{x}), \quad t \ne \tau_k(\boldsymbol{y}) \end{aligned} \tag{5.48}$$

where $\nu_k \in \mathcal{K}$ *and* $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$;

4. $V(t,\boldsymbol{x})$ *is nonincreasing in* t, $\beta(\|\boldsymbol{x}\|) \le V(t, \boldsymbol{x})$ *and for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$,

$$|V(t,\boldsymbol{x}) - V(t,\boldsymbol{y})| \le L_2\|\boldsymbol{x} - \boldsymbol{y}\|;$$

5. $w_{\max}(t, t_0, w_0)$ *be the maximal solution of*

$$\begin{aligned} \dot{w} &= g(t, w), \quad t \notin [\underline{\tau_k}, \overline{\tau_k}], \\ w(\overline{\tau_k}^+) &= \psi_k(w(\underline{\tau_k})), \\ w(t_0^+) &= w_0 \ge 0 \end{aligned} \tag{5.49}$$

where $\underline{\tau_k} = \min(\tau_k^a, \tau_k)$, $\overline{\tau_k} = \max(\tau_k^a, \tau_k)$ *and*

$$\psi(w) = \nu_k[w + L_1L_2K_k\beta^{-1}(w)] + L_1L_2K_k\beta^{-1}(w).$$

Then

$$\begin{aligned} V(t_0^+, \boldsymbol{x}_0 - \boldsymbol{x}_{a0}) \le w_0 \Rightarrow V(t, \boldsymbol{x}(t) - \boldsymbol{x}_a(t)) \le w_{\max}(t, t_0, w_0), \\ t \in [t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\underline{\tau_k}, \overline{\tau_k}]. \end{aligned} \tag{5.50}$$

⊠

Proof. Let us choose $w(t) = V(t, \boldsymbol{x}(t) - \boldsymbol{x}_a(t))$, $w(\overline{\tau_k}^+) = V(\overline{\tau_k}^+, \boldsymbol{x}(\overline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+))$, and $w(t_0^+) = V(t_0^+, \boldsymbol{x}_0 - \boldsymbol{x}_{a0}) \leq w_0$, since V is locally Lipschitzian and from (5.47) we have

$$D^+ w(t) \leq g(t, w(t)), \quad t \neq \underline{\tau_k}, \quad t \neq \overline{\tau_k}. \tag{5.51}$$

Let us consider firstly the case $\tau_k = \underline{\tau_k}$. It follows from conditions 1, 3 and 4 that

$$\begin{aligned}
w(\overline{\tau_k}^+) &= V(\overline{\tau_k}^+, \boldsymbol{x}(\overline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+)) \\
&= V(\overline{\tau_k}^+, \boldsymbol{x}(\overline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+)) - V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+)) \\
&\quad + V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+)) \\
&\leq L_2 \|\boldsymbol{x}(\overline{\tau_k}^+) - \boldsymbol{x}_a(\overline{\tau_k}^+) - \boldsymbol{x}(\underline{\tau_k}^+) + \boldsymbol{x}_a(\overline{\tau_k}^+)\| \qquad \Leftarrow \text{(condition 4)} \\
&\quad + V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\overline{\tau_k}) + U(k, \boldsymbol{x}(\underline{\tau_k})) - U(k, \boldsymbol{x}_a(\overline{\tau_k}))) \\
&\leq L_2 \|\boldsymbol{x}(\overline{\tau_k}^+) - \boldsymbol{x}(\underline{\tau_k}^+)\| \\
&\quad + \nu_k(V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\overline{\tau_k}))) \qquad \Leftarrow (5.48) \\
&\leq \underbrace{L_1 L_2 (\overline{\tau_k} - \underline{\tau_k})}_{\text{condition 1}} + \nu_k(V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\overline{\tau_k}))),
\end{aligned} \tag{5.52}$$

and from condition 4 we have

$$V(\overline{\tau_k}^+, \boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\overline{\tau_k})) \leq L_1 L_2 (\overline{\tau_k} - \underline{\tau_k}) + w(\underline{\tau_k}), \tag{5.53}$$

and from conditions 2 and 4 we have

$$\begin{aligned}
0 \leq \overline{\tau_k} - \underline{\tau_k} &= \tau_k(\boldsymbol{x}_a(\overline{\tau_k})) - \tau_k(\boldsymbol{x}(\underline{\tau_k})) \\
&\leq K_k \|\boldsymbol{x}_a(\overline{\tau_k}) - \boldsymbol{x}(\underline{\tau_k})\| \qquad \Leftarrow \text{(condition 2)} \\
&\leq K_k \beta^{-1}(w(\underline{\tau_k})). \qquad \Leftarrow \text{(condition 4)}
\end{aligned} \tag{5.54}$$

It follows from (5.52), (5.53) and (5.54) that

$$\begin{aligned}
w(\overline{\tau_k}^+) &\leq \nu_k(w(\underline{\tau_k}) + L_1 L_2 K_k \beta^{-1}(w(\underline{\tau_k}))) + L_1 L_2 \beta^{-1}(w(\underline{\tau_k})) \\
&= \psi_k(w(\underline{\tau_k})).
\end{aligned} \tag{5.55}$$

By using the similar process we can also get the same estimation as in (5.55) for the case $\tau_k^a = \overline{\tau_k}$. Therefore, we have the following inequalities:

$$\begin{aligned}
D^+ w(t) &\leq g(t, w(t)), \quad t \notin [\underline{\tau_k}, \overline{\tau_k}], \\
w(\overline{\tau_k}^+) &\leq \psi_k(w(\underline{\tau_k})), \\
w(t_0^+) &\leq w_0.
\end{aligned} \tag{5.56}$$

Hence, it follows Theorem 5.2.1 that we finish the proof. ■

Corollary 5.2.1. *Let us suppose that the following conditions hold:*

1. *$\boldsymbol{f}(t,0)+\boldsymbol{u}(t,0)=0$ and $U(k,0)=0$ for all $k\in\mathbb{N}$;*
2. *$V(t,\boldsymbol{x}(t))\in C[\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}_+]$ is locally Lipschitzian in $\boldsymbol{x}$ and*

$$\begin{aligned} D^+V(t,\boldsymbol{x}(t)) &\le g(t,V(t,\boldsymbol{x}(t))), \quad t\ne\tau_k(\boldsymbol{x}),\\ V(t,\boldsymbol{x}+U(k,\boldsymbol{x})) &\le \psi_k(V(t,\boldsymbol{x})), \quad t=\tau_k(\boldsymbol{x}) \end{aligned} \tag{5.57}$$

   where $g\in C[\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}]$ and $\psi_k\in C[\mathbb{R}_+,\mathbb{R}_+]$ is nondecreasing;
3. *$\boldsymbol{x}(t)=\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ is any a solution of system (5.41) such that $V(t_0^+,\boldsymbol{x}_0)\le w_0$ for $t\ge t_0$. $\boldsymbol{x}(t)$ hits the switching surfaces Σ_k at $t=\tau_k$. $w_{\max}(t)$ is the maximal solution of the comparison system (3.3) for $t\ge t_0$.*

Then $V(t,\boldsymbol{x}(t))\le w_{\max}(t)$ for $t\ge t_0$. ⊠

Proof. From condition 1 we know that the trivial solution of impulsive control system (5.41) exists. Then let us choose the prescribed solution of (5.41) as $\boldsymbol{x}_a(t)=0$, it follows Theorem 5.2.2 that we finish the proof. ■

From Corollary 5.2.1 we then have the following corollary.

Corollary 5.2.2. *If in Corollary 5.2.1 we choose*

1. *$g(t,w)=0$ and $\psi_k(w)=w$ for all $k\in\mathbb{N}$, then $V(t,\boldsymbol{x}(t))$ is nonincreasing in t and $V(t,\boldsymbol{x}(t))\le V(t_0^+,\boldsymbol{x}_0)$ for $t\ge t_0$;*
2. *$g(t,w)=0$ and $\psi_k(w)=d_kw$, $d_k\ge 0$, then we have for $t\ge t_0$*

$$V(t,\boldsymbol{x}(t))\le V(t_0^+,\boldsymbol{x}_0)\prod_{t_0<\tau_k<t} d_k.$$

⊠

Theorem 5.2.3. *Let us suppose that the following conditions hold:*

1. *$h_0\in\mathcal{H}_0$, $h\in\mathcal{H}_0$ and h_0 is finer than h;*
2. *$V(t,\boldsymbol{x})\in\mathcal{V}_2$ is locally Lipschitzian in $\boldsymbol{x}$ on every $\mathfrak{G}_i$, h-positive definite on $\mathcal{S}_\rho(h)$ and*

$$D^+V(t,\boldsymbol{x})\le 0 \ \text{on}\ \mathfrak{G}\cap\mathcal{S}_\rho(h);$$

3. *$V(t^+,\boldsymbol{x}+U(k,\boldsymbol{x}))\le V(t,\boldsymbol{x})$ on $\mathfrak{G}_k\cap\mathcal{S}_\rho(h)$.*

Then the trivial solution of the impulsive control system (5.41) is

1. *(h_0,h)-stable if $V(t,\boldsymbol{x})$ is weakly h_0-decrescent;*
2. *(h_0,h)-uniformly stable if $V(t,\boldsymbol{x})$ is h_0-decrescent.*

⊠

Proof. Let us first prove conclusion 1. Since $V(t,\boldsymbol{x})$ is weakly h_0-decrescent, there is a $\varepsilon_0>0$ and a function $\alpha\in\mathcal{CK}$ such that

$$h_0(t,\boldsymbol{x})<\varepsilon_0\Rightarrow V(t,\boldsymbol{x})\le\alpha(t,h_0(t,\boldsymbol{x})). \tag{5.58}$$

From the assumption that $V(t, \boldsymbol{x})$ is h-positive definite on $S_\rho(h)$, we know that there is a $\beta \in \mathcal{K}$ such that

$$\beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in S_\rho(h). \tag{5.59}$$

It follows from condition 1 that there are a $\varepsilon_1 > 0$ and a $\kappa \in \mathcal{K}$ such that

$$h_0(t, \boldsymbol{x}) < \varepsilon_1 \Rightarrow h(t, \boldsymbol{x}) \leq \kappa(h_0(t, \boldsymbol{x})). \tag{5.60}$$

From the assumptions on α and κ, given $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there are $\varepsilon_2 \in (0, \varepsilon_0)$ and $\varepsilon_3 \in (0, \varepsilon_1)$ such that

$$\alpha(t_0, \varepsilon_2) < \beta(\eta), \quad \kappa(\varepsilon_3) < \rho. \tag{5.61}$$

Let us choose $\varepsilon = \min(\varepsilon_2, \varepsilon_3)$. Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be any a solution of system (5.41) with $h_0(t_0, \boldsymbol{x}_0) < \varepsilon$, then from Corollary 5.2.2 we know that $V(t, \boldsymbol{x}(t))$ is nonincreasing. Then it follows from (5.58), (5.59) and (5.61) we have for $t \geq t_0$

$$\underbrace{\beta(h(t, \boldsymbol{x})) \leq V(t,}_{(5.59)}\underbrace{\boldsymbol{x}(t)) \leq V(t_0^+,}_{\text{nonincreasing}}\underbrace{\boldsymbol{x}_0) \leq \alpha(t_0,}_{(5.58)}\underbrace{h_0(t_0, \boldsymbol{x}_0)) < \beta(\eta)}_{(5.61)} \tag{5.62}$$

from which we have

$$h_0(t_0, \boldsymbol{x}_0) < \varepsilon \Rightarrow h(t, \boldsymbol{x}) < \eta \text{ for } t \geq t_0. \tag{5.63}$$

Therefore, the trivial solution of the impulsive control system (5.41) is (h_0, h)-stable.

Next, let us prove conclusion 2. Since $V(t, \boldsymbol{x})$ is h_0-decrescent, (5.58) and (5.61) hold with $\alpha \in \mathcal{K}$ independent of t. Then we can choose ε independent of t_0 such that (5.63) holds for $h_0(t_0, \boldsymbol{x}_0) < \varepsilon$. Therefore, the trivial solution of the impulsive control system (5.41) is (h_0, h)-uniformly stable. ■

Theorem 5.2.4. *Let us suppose that the following conditions are satisfied:*

1. $h_0 \in \mathcal{H}_0$, $h \in \mathcal{H}_0$ *and* h_0 *is finer than* h*;*
2. $V(t, \boldsymbol{x}) \in \mathcal{V}_2$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, *h-positive definite on* $\mathcal{S}_\rho(h)$, *weakly h_0-decrescent and*

   $$D^+V(t, \boldsymbol{x}) \leq -\gamma(t)\zeta(V_1(t, \boldsymbol{x})) \text{ on } \mathfrak{G} \cap \mathcal{S}_\rho(h)$$

   where $\zeta \in \mathcal{K}$, $V_1 \in \mathcal{V}_2$ *and* $\gamma(t)$ *is integrally positive;*
3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x})$ *on* $\Sigma_k \cap \mathcal{S}_\rho(h)$*;*
4. $V_1(t, \boldsymbol{x})$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, *h-positive definite and for every piecewise continuous function* $\boldsymbol{p}(t)$ *with discontinuities at* $\tau_k = \tau_k(\boldsymbol{p}(t))$, $k \in \mathbb{N}$, *such that for* $(t, \boldsymbol{p}(t)) \in \mathcal{S}_\rho(h)$

   $$\int_0^t [D^+V_1(s, \boldsymbol{p}(s))]_+ ds \quad (\text{resp. } \int_0^t [D^+V_1(s, \boldsymbol{p}(s))]_- ds)$$

is uniformly continuous on $\mathbb{R}_+$ *and*

$$V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V_1(t, \boldsymbol{x}) \quad (\text{resp. } V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq V_1(t, \boldsymbol{x}))$$

on $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then the trivial solution of the impulsive control system (5.41) is (h_0, h)*-asymptotically stable.* ⊠

Proof. It follows from Theorem 5.2.3 that the trivial solution of the impulsive control system (5.41) is (h_0, h)-stable.

Given a $\rho > 0$ and $t_0 \in \mathbb{R}_+$, there is a $\varepsilon_0 = \varepsilon_0(t_0, \rho) > 0$ such that for any solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of system (5.41) $h_0(t_0, \boldsymbol{x}_0) < \varepsilon_0$ implies

$$h(t, \boldsymbol{x}(t)) < \rho, \quad t \geq t_0. \tag{5.64}$$

We then have the following claim:
*Claim 5.2.4*a:

$$\lim_{t\to\infty} \inf V_1(t, \boldsymbol{x}(t)) = 0. \tag{5.65}$$

If Claim 5.2.4a is false, then there is a $\theta > 0$ such that for some $T > 0$

$$V_1(t, \boldsymbol{x}(t)) \geq \theta \text{ for } t \geq t_0 + T. \tag{5.66}$$

Let us choose a sequence

$$t_0 + T < a_1 < b_1 < \cdots < a_k < b_k < \cdots$$

such that $b_k - a_k \geq \theta$ for $k \in \mathbb{N}$. Then from (5.66) and condition 2 we have the following contradiction:

$$\begin{aligned}
\lim_{t\to\infty} V(t, \boldsymbol{x}(t)) &\leq V(t_0, \boldsymbol{x}_0) - \int_{t_0}^{\infty} \gamma(s)\zeta(V_1(s, \boldsymbol{x}(s)))ds \quad &\Leftarrow \text{(condition 2)} \\
&\leq V(t_0, \boldsymbol{x}_0) - \zeta(\theta) \int_{\bigcup_{k=1}^{\infty}[a_k, b_k]} \gamma(s)ds \quad &\Leftarrow (5.66) \\
&= -\infty.
\end{aligned} \tag{5.67}$$

Therefore, Claim 5.2.4a is true and (5.65) holds.

We then have the following claim:
*Claim 5.2.4*b:

$$\lim_{t\to\infty} \sup V_1(t, \boldsymbol{x}(t)) = 0. \tag{5.68}$$

If Claim 5.2.4b is false, then let us suppose that

$$\lim_{t\to\infty} \sup V_1(t, \boldsymbol{x}(t)) > 0$$

then there is a $\nu > 0$ such that

$$\lim_{t\to\infty} \sup V_1(t, \boldsymbol{x}(t)) > 2\nu.$$

Let condition 4 hold with $[\cdot]_+$ and since (5.65) is true, we can find a sequence

$$t_0 < t_1^a < t_1^b < \cdots < t_k^a < t_k^b < \cdots$$

such that for $k = 1, 2, \cdots$, we have

$$V_1(t_k^a, \boldsymbol{x}(t_k^a)) = \nu \text{ and } V_1(t_k^b, \boldsymbol{x}(t_k^b)) = 2\nu. \tag{5.69}$$

It follows from $V_1(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) \leq V_1(\tau_k, \boldsymbol{x}(\tau_k))$ and (5.69) that there is a sequence

$$t_0 < s_1^a < s_1^b < \cdots < s_k^a < s_k^b < \cdots$$

such that for $k = 1, 2, \cdots$, we have $t_k^a \leq s_k^a \leq s_k^b \leq t_k^b$ and

$$\begin{aligned} &V_1(s_k^a, \boldsymbol{x}(s_k^a)) = \nu, \quad V_1(s_k^b, \boldsymbol{x}(s_k^b)) = 2\nu \text{ and} \\ &\nu \leq V_1(t, \boldsymbol{x}(t)) \leq 2\nu \text{ for } t \in [s_k^a, s_k^b] \end{aligned} \tag{5.70}$$

from which we have

$$\begin{aligned} 0 < \nu &< V_1(s_k^b, \boldsymbol{x}(s_k^b)) - V_1(s_k^a, \boldsymbol{x}(s_k^a)) \\ &\leq \int_{s_k^a}^{s_k^b} [D^+ V_1(s, \boldsymbol{x}(s))]_+ ds, \quad k = 1, 2, \cdots . \end{aligned} \tag{5.71}$$

Then in view of condition 4, we have for some $\xi > 0$

$$s_k^b - s_k^a \geq \xi, \quad k = 1, 2, \cdots . \tag{5.72}$$

Therefore, from (5.70), (5.72) and condition 2 we have the following contradiction:

$$\begin{aligned} \lim_{t\to\infty} V(t, \boldsymbol{x}(t)) &\leq V(t_0, \boldsymbol{x}(t_0)) - \int_{t_0}^{\infty} \gamma(s)\zeta(V_1(s, \boldsymbol{x}(s)))ds \Leftarrow \text{(Condition 2)} \\ &\leq \underbrace{V(t_0, \boldsymbol{x}(t_0)) - \zeta(\nu) \int_{\bigcup_{k=1}^{\infty}[s_k^a, s_k^b]} \gamma(s)ds}_{(5.70)\&(5.72)} \\ &= -\infty. \end{aligned} \tag{5.73}$$

Therefore Claim 5.2.4b is true. It follows from Claims 5.2.4a and 5.2.4b that

$$\lim_{t\to\infty} V_1(t, \boldsymbol{x}(t)) = 0$$

from which and since $V_1(t, \boldsymbol{x}(t))$ is h-positive definite, we have

$$\lim_{t\to\infty} h(t, \boldsymbol{x}(t)) = 0.$$

Therefore we finish the proof. ■

We then have the following corollaries.

Corollary 5.2.3. *Let us suppose that the following conditions are satisfied:*

1. $h_0 \in \mathcal{H}_0$, $h \in \mathcal{H}_0$ *and* h_0 *is finer than* h;
2. $V(t, \boldsymbol{x}) \in \mathcal{V}_2$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, h*-positive definite on* $\mathcal{S}_\rho(h)$, *weakly* h_0*-decrescent and*
$$D^+V(t, \boldsymbol{x}) \le -\gamma(t)\zeta(V_1(t, \boldsymbol{x})) \text{ on } \mathfrak{G} \cap \mathcal{S}_\rho(h)$$
where $\zeta \in \mathcal{K}$, $V_1 \in \mathcal{V}_2$ *and* $\gamma(t)$ *is integrally positive;*
3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(t, \boldsymbol{x})$ *on* $\Sigma_k \cap \mathcal{S}_\rho(h)$;
4. $V_1(t, \boldsymbol{x})$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, h*-positive definite and* $D^+V_1(t, \boldsymbol{x})$ *is bounded from above (resp. from below) on* $\mathfrak{G} \cap \mathcal{S}_\rho(h)$ *and*
$$V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V_1(t, \boldsymbol{x}) \quad (\text{resp. } V_1(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \ge V_1(t, \boldsymbol{x}))$$
on $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then the trivial solution of the impulsive control system (5.41) is (h_0, h)*-asymptotically stable.* ⊠

Corollary 5.2.4. *Let us suppose that the following conditions are satisfied:*

1. $h_0 \in \mathcal{H}_0$, $h \in \mathcal{H}_0$ *and* h_0 *is finer than* h;
2. $V(t, \boldsymbol{x}) \in \mathcal{V}_2$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, h*-positive definite on* $\mathcal{S}_\rho(h)$, *weakly* h_0*-decrescent and*
$$D^+V(t, \boldsymbol{x}) \le -\gamma(t)\zeta(V(t, \boldsymbol{x})) \text{ on } \mathfrak{G} \cap \mathcal{S}_\rho(h)$$
where $\zeta \in \mathcal{K}$ *and* $\gamma(t)$ *is integrally positive;*
3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(t, \boldsymbol{x})$ *on* $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then the trivial solution of the impulsive control system (5.41) is (h_0, h)*-asymptotically stable.* ⊠

Theorem 5.2.5. *Let us suppose that the following conditions are satisfied:*

1. $h_0 \in \mathcal{H}_0$, $h \in \mathcal{H}_0$ *and* h_0 *is finer than* h;
2. $V(t, \boldsymbol{x}) \in \mathcal{V}_2$ *is locally Lipschitzian in* $\boldsymbol{x}$ *on* $\mathfrak{G}_k$, $k \in \mathbb{N}$, h*-positive definite on* $\mathcal{S}_\rho(h)$, h_0*-decrescent and*
$$D^+V(t, \boldsymbol{x}) \le -\gamma(t)\zeta(h_0(t, \boldsymbol{x})) \text{ on } \mathfrak{G} \cap \mathcal{S}_\rho(h)$$
where $\zeta \in \mathcal{K}$ *and* $\gamma(t)$ *is integrally positive;*
3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(t, \boldsymbol{x})$ *on* $\Sigma_k \cap \mathcal{S}_\rho(h)$.

Then the trivial solution of the impulsive control system (5.41) is (h_0, h)*-uniformly asymptotically stable.* ⊠

Proof. From Theorem 5.2.3 we know that the trivial solution of the impulsive control system (5.41) is (h_0, h)-uniformly stable. Therefore, for $\rho > 0$ there is a $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that for any solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of system (5.41) we have

$$h_0(t_0, \boldsymbol{x}_0) < \varepsilon_0 \Rightarrow h(t, \boldsymbol{x}(t)) < \rho \text{ for } t \geq t_0. \tag{5.74}$$

For a given $\eta \in (0, \rho)$, let $\varepsilon = \varepsilon(\eta)$, then we have the following claim:
Claim 5.2.5: There is a $T = T(\eta) > 0$ such that, for some $t_1 \in [t_0, t_0 + T]$, we have

$$h_0(t_1, \boldsymbol{x}(t_1)) < \varepsilon. \tag{5.75}$$

If Claim 5.2.5 is false, then for any $T > 0$ there is a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of system (5.41) satisfying (5.74) such that

$$h_0(t, \boldsymbol{x}(t)) \geq \varepsilon, \quad t \in [t_0, t_0 + T]. \tag{5.76}$$

From conditions 2 and 3 we know that $V(t, \boldsymbol{x}(t))$ is nonincreasing and from condition 2 we know that there is an $\alpha \in \mathcal{K}$ such that, for every $\boldsymbol{x}(t)$ satisfying (5.74),

$$\int_{t_0}^{\infty} \gamma(s)\zeta(h_0(s, \boldsymbol{x}(s)))ds \leq \alpha(\varepsilon_0). \tag{5.77}$$

From the assumption on $\gamma(t)$ we know that there is a $T_1 > 0$ such that

$$\int_{t_0}^{t_0+T_1} \gamma(s)ds > \frac{\alpha(\varepsilon_0) + 1}{\zeta(\varepsilon)}. \tag{5.78}$$

Let $\boldsymbol{x}(t)$ satisfy (5.76), then from (5.77) and (5.78) we have the following contradiction:

$$\begin{aligned}
\alpha(\varepsilon_0) &\geq \int_{t_0}^{\infty} \gamma(s)\zeta(h_0(s, \boldsymbol{x}(s)))ds && \Leftarrow (5.77) \\
&> \zeta(\varepsilon) \int_{t_0}^{\infty} \gamma(s)ds && \Leftarrow (5.76) \\
&> \zeta(\varepsilon) \int_{t_0}^{t_0+T_1} \gamma(s)ds \\
&> \alpha(\varepsilon_0) + 1. && \Leftarrow (5.78)
\end{aligned} \tag{5.79}$$

Therefore, the trivial solution of the impulsive control system (5.41) is (h_0, h)-uniformly asymptotically stable. ■

We then study the case when in the impulsive control system (5.41), the constrains on $\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})$ are relaxed as follows. Suppose that $\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})$ is continuous on each $\mathfrak{G}_i$ and for $(t, \boldsymbol{y}) \in \mathfrak{G}_{k+1}$ and each $(\tau_k, \boldsymbol{x}) \in \Sigma_k$, $k \in \mathbb{N}$, the following limit exists:

$$\lim_{(t,\boldsymbol{y}) \to (\tau_k, \boldsymbol{x})} \boldsymbol{f}(t, \boldsymbol{y}) + \boldsymbol{u}(t, \boldsymbol{y}) = \boldsymbol{f}(\tau_k^+, \boldsymbol{y}) + \boldsymbol{u}(\tau_k^+, \boldsymbol{y}).$$

Theorem 5.2.6. *Let us suppose that the following conditions are satisfied:*

1. $h_0 \in \mathcal{H}_0$, $h \in \mathcal{H}$, $V(t, \boldsymbol{x}) \in \mathcal{V}_0$ *is h-positive definite on* $\mathcal{S}_\rho(h)$ *and for* $(t, \boldsymbol{x}) \in \mathfrak{G} \cap \mathcal{S}_\rho(h)$
$$D^+V(t, \boldsymbol{x}) \leq 0;$$
2. *for all* $(\tau_k, \boldsymbol{x}) \in \Sigma_k \cap \mathcal{S}_\rho(h)$, $V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(\tau_k, \boldsymbol{x})$*;*
3. *there is an* $\eta \in (0, \rho)$ *such that* $h(t, \boldsymbol{x}) < \eta$ *implies* $h(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) < \rho$.

Then, the impulsive control system (5.41) is

1. (h_0, h)*-stable if* h_0 *is finer than* h *and* $V(t, \boldsymbol{x})$ *is* h_0*-weakly decrescent;*
2. (h_0, h)*-uniformly stable if* h_0 *is uniformly finer than* h *and* $V(t, \boldsymbol{x})$ *is* h_0*-decrescent.*

⊠

Proof. Let us prove conclusion 1 first. The assumption that $V(t, \boldsymbol{x})$ is h_0-weakly decrescent leads to the fact that there are a $\varepsilon_0 > 0$ and an $\alpha \in \mathcal{CK}$ such that

$$h_0(t, \boldsymbol{x}) < \varepsilon_0 \Rightarrow \; V(t, \boldsymbol{x}) \leq \alpha(t, h_0(t, \boldsymbol{x})). \tag{5.80}$$

From the assumptions we know that there are $\beta \in \mathcal{K}$, $\gamma \in \mathcal{CK}$ and $\varepsilon_1 > 0$ such that

$$\beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \mathcal{S}_\rho(h), \tag{5.81}$$

and for $h_0(t, \boldsymbol{x}) < \varepsilon_1$ we have

$$h(t, \boldsymbol{x}) \leq \gamma(t, h_0(t, \boldsymbol{x})). \tag{5.82}$$

Let us choose $t_0 \in \mathbb{R}_+$ and $\xi \in (0, \eta)$, then there are $\varepsilon_2 = \varepsilon_2(t_0, \xi)$ and $\varepsilon_3 = \varepsilon_3(t_0, \rho)$ such that $\varepsilon_2 \in (0, \varepsilon_0)$, $\varepsilon_3 \in (0, \varepsilon_1)$ and

$$\alpha(t_0, \varepsilon_2) < \beta(\xi), \quad \gamma(t_0, \varepsilon_3) < \rho. \tag{5.83}$$

Let us define $\varepsilon = \min(\varepsilon_2, \varepsilon_3)$, then it follows from (5.80) and (5.83) that $h_0(t_0, \boldsymbol{x}_0) < \xi$ implies

$$\beta(h(t_0, \boldsymbol{x}_0)) \leq V(t_0, \boldsymbol{x}_0) \leq \alpha(t_0, h_0(t_0, \boldsymbol{x}_0)) < \beta(\xi). \tag{5.84}$$

From (5.84) and the assumption on β we have

$$h(t_0, \boldsymbol{x}_0) < \xi.$$

Then we have the following claim:
Claim 5.2.6: Given any a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of impulsive control system (5.41), we have for $t \geq t_0$

$$h_0(t_0, \boldsymbol{x}_0) < \varepsilon \Rightarrow h(t, \boldsymbol{x}(t)) < \xi. \tag{5.85}$$

If Claim 5.2.6 is false, then there is a solution, $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$, of impulsive control system (5.41) with $h_0(t_0, \boldsymbol{x}_0) < \varepsilon$ and $t_1 \in (\tau_k, \tau_{k+1})$ for some k such that

$$\begin{aligned} &h(t_1, \boldsymbol{x}_1(t_1)) \geq \xi \text{ and} \\ &h(t, \boldsymbol{x}_1(t)) < \xi \text{ for } t \in [t_0, \tau_k]. \end{aligned} \tag{5.86}$$

It follows from assumption 3 and $\xi \in (0, \rho_0)$ that

$$h(\tau_k^+, \boldsymbol{x}_1(\tau_k^+)) = h(\tau_k^+, \boldsymbol{x}_1(\tau_k) + U(k, \boldsymbol{x}_1(\tau_k))) < \rho \tag{5.87}$$

because $h(\tau_k, \boldsymbol{x}_1(\tau_k)) < \xi$ by (5.86). We then can find a $t_2 \in (\tau_k, t_1]$ such that

$$\begin{aligned} &\rho > h(t_2, \boldsymbol{x}_1(t_2)) \geq \xi \text{ and} \\ &h(t, \boldsymbol{x}_1(t)) < \rho \text{ for } t \in [t_0, t_2). \end{aligned} \tag{5.88}$$

Let $w(t) = V(t, \boldsymbol{x}_1(t))$ for $t \in [t_0, t_2]$ and from assumptions 1 and 2 we have

$$\begin{aligned} &D^+ w(t) \leq 0, \quad t \neq \tau_i, \quad t \in [t_0, t_2], \\ &w(\tau_i^+) \leq w(\tau_i), \quad i = 1, 2, \cdots, k. \end{aligned} \tag{5.89}$$

Therefore $V(t, \boldsymbol{x}_1(t))$ is nonincreasing on $[t_0, t_2]$. Then from (5.81), (5.83) and (5.88) we have the following contradiction:

$$\beta(\xi) \leq \beta(h(t_2, \boldsymbol{x}_1(t_2))) \leq V(t_2, \boldsymbol{x}_1(t_2)) \leq V(t_0, \boldsymbol{x}_0) < \beta(\xi). \tag{5.90}$$

Therefore Claim 5.2.6 is true and the impulsive control system (5.41) is (h_0, h)-stable.

The proof of conclusion 2 can be performed in a similar way. ■

Theorem 5.2.7. *Let us suppose that the following conditions are satisfied:*

1. *$h_0 \in \mathcal{H}$, $h \in \mathcal{H}$ and h_0 is finer that h.*
2. *$V(t, \boldsymbol{x}) \in \mathcal{V}_0$ is h_0-weakly decrescent , h-positive definite on $\mathcal{S}_\rho(h)$ and for $(t, \boldsymbol{x}) \in \mathfrak{G} \cap \mathcal{S}_\rho(h)$*
$$D^+ V(t, \boldsymbol{x}) \leq 0,$$
and
$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) - V(t, \boldsymbol{x}) \leq -\zeta_k \psi(V(t, \boldsymbol{x})), \quad (t, \boldsymbol{x}) \in \Sigma_k \cap \mathcal{S}_\rho(h)$$
where $\zeta_k \geq 0, \sum_{i=1}^{\infty} \zeta_i = \infty$, $\psi \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi(0) = 0$ and $\psi(w) > 0$ for $w > 0$;
3. *there is an $\eta \in (0, \rho)$ such that $h(t, \boldsymbol{x}) < \eta$ implies $h(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) < \rho$;*

Then, the impulsive control system (5.41) is (h_0, h)-asymptotically stable. ⊠

Proof. From condition 2 we have

$$V(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \le V(\tau_k, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in \Sigma_k \cap \mathcal{S}_\rho(h),$$

from which and Theorem 5.2.6 we know that system (5.41) is (h_0, h)-stable. Therefore, for any a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ with $h_0(t_0, \boldsymbol{x}_0) < \varepsilon_0$, of (5.41), $t_0 \in \mathbb{R}_+$ and $\rho > 0$, there is a $\varepsilon_0 = \varepsilon_0(t_0, \rho) > 0$ such that $h_0(t_0, \boldsymbol{x}_0) < \varepsilon_0$ implies, for $t \ge t_0$

$$h(t, \boldsymbol{x}(t)) < \rho.$$

Let us define $w(t) = V(t, \boldsymbol{x}(t))$, then from the assumptions we know that $w(t)$ is nonincreasing and bounded from below, therefore the following limit exists:

$$\lim_{t\to\infty} w(t) = \varpi.$$

We then have the following claim:
Claim 5.2.7: $\varpi = 0$.

If Claim 5.2.7 is false, then there is a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of (5.41) such that $\varpi < 0$. Let us assume that $\boldsymbol{x}(t)$ hits the switching surfaces Σ_k at τ_k, $k \in \mathbb{N}$ and let

$$\theta = \min_{\varpi \le s \le w(t_0)} \psi(s).$$

Then it follows from assumption 2 that

$$w(\tau_k^+) - w(\tau_k) \le -\zeta_k \psi(w(\tau_k)) \le -\theta\zeta_k, \quad k \in \mathbb{N} \tag{5.91}$$

from which we have

$$w(\tau_k^+) \le w(t_0^+) - \theta \sum_{i=1}^{k} \zeta_i, \quad k \in \mathbb{N} \tag{5.92}$$

It follows from (5.92) and the assumption of $\sum_{i=1}^{\infty} \zeta_i = \infty$ that

$$\lim_{k\to\infty} w(\tau_k^+) = -\infty$$

which is a contradiction. Therefore, Claim 5.2.7 is true and we have

$$\lim_{t\to\infty} h(t, \boldsymbol{x}(t)) = 0$$

from which we know that the impulsive control system (5.41) is (h_0, h)-asymptotically stable. ■

5.3 Stability of Prescribed Control Strategies

Let us consider the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}), & t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & k = 1, 2, \cdots. \end{cases} \tag{5.93}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is the nonlinearity of the uncontrolled plant, $\boldsymbol{u} \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is the continuous control input, $U(k, \boldsymbol{x}) \in C[\mathbb{N} \times \mathbb{R}^n, \mathbb{R}^n]$ is the impulsive control input, $\tau_k \in C^1[\mathbb{R}^n, (0, \infty)]$, $\tau_k(\boldsymbol{x}) < \tau_{k+1}(\boldsymbol{x})$ for all k and $\lim_{k\to\infty} \tau_k(\boldsymbol{x}) = \infty$ for every $\boldsymbol{x} \in \mathbb{R}^n$. Let us assume that the solutions $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (5.93) exist and are unique for $t \geq t_0$, and each solution of (5.93) hits any given switching surface $\Sigma_k : t = \tau_k(\boldsymbol{x})$ exactly once.

Theorem 5.3.1. *Let $\boldsymbol{x}_a(t) = \boldsymbol{x}_a(t, t_0, \boldsymbol{x}_{a0})$ be any given solution of system (5.93) on $[t_0, \infty)$ which hits the switching surfaces at moments τ_k^a, $k \in \mathbb{N}$, and let us suppose that the following conditions are satisfied:*

1. *$\|\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})\| \leq L_1$, $L_1 > 0$ and $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$;*
2. *for $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$ and for all k, we have*
$$\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}} [\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})] \leq 0 \text{ and } |\tau_k(\boldsymbol{x}) - \tau_k(\boldsymbol{y})| \leq K_k \|\boldsymbol{x} - \boldsymbol{y}\|$$
where $K_k \geq 0$ and $\sum_{k=1}^{\infty} K_k$ converges;
3. *$V(t, \boldsymbol{x}) \in C[\mathbb{R}_+ \times \mathcal{S}_\rho, \mathbb{R}_+]$ is nonincreasing in t for fixed $\boldsymbol{x}$ and there is an $L_2 > 0$ such that*
$$|V(t, \boldsymbol{x}) - V(t, \boldsymbol{y})| \leq L_2 \|\boldsymbol{x} - \boldsymbol{y}\|$$
where $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$ and $(t, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$;
4. *for $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$ and $(t, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$ and $k \in \mathbb{N}$*
$$V(t, \boldsymbol{x} - \boldsymbol{y} + U(k, \boldsymbol{x}) - U(k, \boldsymbol{y})) \leq V(t, \boldsymbol{x} - \boldsymbol{y});$$
5. *for $\boldsymbol{x} \in \mathcal{S}_\rho$ and for $k \in \mathbb{N}$, $\|U(k, \boldsymbol{x})\| \leq \rho/3$;*
6. *$V(t, 0) = 0$ and there is an $L_3 > 0$ such that*
$$L_3 \|\boldsymbol{x} - \boldsymbol{x}_a(t)\| \leq V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)), \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)};$$
7. *$D^+ V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)) \leq 0$, $t \neq \tau_k(\boldsymbol{x})$, $t \neq \tau_k^a$, $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$.*

Then $\boldsymbol{x}_a(t)$ is uniformly stable. ⊠

Proof. From conditions 3 and 6 we have

$$\underbrace{V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)) = V(t,}_{\text{condition 6}} \boldsymbol{x} - \boldsymbol{x}_a(t)) \underbrace{-V(t, 0) \leq L_2 \|\boldsymbol{x}}_{\text{condition 3}} - \boldsymbol{x}_a(t)\| \tag{5.94}$$

from which we have, for $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$

$$L_3\|\boldsymbol{x} - \boldsymbol{x}_a(t)\| \underbrace{\leq}_{\text{condition 6}} V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)) \leq L_2\|\boldsymbol{x} - \boldsymbol{x}_a(t)\|. \tag{5.95}$$

Because $\sum_{k=1}^\infty K_k$ converges we know that the sequence $\{K_k\}_{k=1}^\infty$ is bounded and the infinite product $\prod_{k=1}^\infty(1+2L_1L_2K_k/L_3)$ converges. Hence, there are two constants $K > 0$ and $L_4 > 0$ such that $K_k \in [0, K]$ and

$$\prod_{k=1}^{\infty}\left(1 + \frac{2L_1L_2K_k}{L_3}\right) \leq L_4.$$

Let us set $\eta \in (0, \rho/3)$, $\xi > 0$ and $t_0 \in \mathbb{R}_+$ and choose a $\varepsilon = \varepsilon(\eta, \xi) > 0$ such that

$$\varepsilon = \min\left(\frac{\eta}{L_4+1}, \frac{L_3\xi}{L_2(2KL_4+1)}\right). \tag{5.96}$$

Let us suppose that $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is any a solution of system (5.93) satisfying $\|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| < \varepsilon$. Let us also suppose that $\boldsymbol{x}(t)$ hits the switching surfaces Σ_k at the moments τ_k^0. Then we have the following claim:
Claim 5.3.1a:

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \eta, \quad t \geq t_0, \quad t \notin (\underline{\tau_k}, \overline{\tau_k}], \quad k \in \mathbb{N} \tag{5.97}$$

where $\underline{\tau_k} = \min(\tau_k^a, \tau_k^0)$ and $\overline{\tau_k} = \max(\tau_k^a, \tau_k^0)$.

If Claim 5.3.1a is not true, then there is a $t_1 \in (\overline{\tau_k}, \underline{\tau_{k+1}}]$ for some fixed k satisfying

$$\begin{aligned}
&\|\boldsymbol{x}(t_1) - \boldsymbol{x}_a(t_1)\| = \eta_1 \geq \eta \text{ and} \\
&\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \eta \text{ for } t \in [t_0, \underline{\tau_k}] \setminus \bigcup_{i=1}^{k}(\underline{\tau_i}, \overline{\tau_i}].
\end{aligned} \tag{5.98}$$

Let us first consider the case when $\underline{\tau_i} = \tau_i^0$. In this case we have, in view of condition 5 and (5.98)

$$\begin{aligned}
\|\boldsymbol{x}(\underline{\tau_i}^+) - \boldsymbol{x}_a(\underline{\tau_i}^+)\| &\leq \|\boldsymbol{x}(\underline{\tau_i}^+) - \boldsymbol{x}_a(\underline{\tau_i})\| + \|U(i, \boldsymbol{x}(\tau_i)\| \\
&\leq \eta + \frac{\rho}{3} \\
&< \frac{2\rho}{3}, \quad 1 \leq i \leq k. \qquad \Leftarrow [\eta \in (0, \rho/3)]
\end{aligned} \tag{5.99}$$

Similarly, in the case when $\overline{\tau_i} = \tau_i^0$ we have the following estimate

$$\|\boldsymbol{x}(\overline{\tau_i}^+) - \boldsymbol{x}_a(\overline{\tau_i}^+)\| < \frac{2\rho}{3}, \quad 1 \leq i \leq k. \tag{5.100}$$

We then have the following claim:
Claim 5.3.1b:

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \frac{2\rho}{3}, \quad t \in (\underline{\tau_i}, \overline{\tau_i}], \quad 1 \le i \le k. \tag{5.101}$$

If Claim 5.3.1b is not true, then let us suppose that there is $t_2 \in (\underline{\tau_i}, \overline{\tau_i}]$ such that

$$\begin{aligned} &\|\boldsymbol{x}(t_2) - \boldsymbol{x}_a(t_2)\| = \frac{2\rho}{3} \text{ and} \\ &\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \frac{2\rho}{3} \text{ for } t \in (\underline{\tau_i}, t_2). \end{aligned} \tag{5.102}$$

For $t \in (\underline{\tau_i}, t_2]$ let us use the notation $w(t) = V(t, \boldsymbol{x}(t) - \boldsymbol{x}_a(t))$, then it follows from condition 7 that

$$D^+ w(t) \le 0, \quad t \in (\underline{\tau_i}, t_2]. \tag{5.103}$$

Then it follows from (5.95), (5.99), (5.102) and (5.103) that

$$\begin{aligned} \frac{2\rho L_3}{3} &= L_3\|\boldsymbol{x}(t_2) - \boldsymbol{x}_a(t_2)\| && \Leftarrow (5.102) \\ &\le V(t_2, \boldsymbol{x}(t_2) - \boldsymbol{x}_a(t_2)) && \Leftarrow (5.95) \\ &\le V(\underline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}^+) - \boldsymbol{x}_a(\underline{\tau_i}^+)) && \Leftarrow (5.103) \\ &\le L_2\|\boldsymbol{x}(\underline{\tau_i}^+) - \boldsymbol{x}_a(\underline{\tau_i}^+)\| && \Leftarrow \text{(condition 3)} \\ &< \frac{2\rho L_2}{3} && \Leftarrow (5.99) \end{aligned} \tag{5.104}$$

From (5.95) we know $L_3 \le L_2$, therefore (5.104) leads to a contradiction. Thus, Claim 5.3.1b is true.

It follows from (5.101) and condition 5 that

$$\|\boldsymbol{x}(\overline{\tau_k}^+) - \underbrace{\boldsymbol{x}_a(\overline{\tau_k}^+)\| \le \|\boldsymbol{x}(\overline{\tau_k})}_{\text{condition 5}} \underbrace{-\boldsymbol{x}_a(\overline{\tau_k})\| + \frac{\rho}{3} < \rho}_{(5.101)}$$

from which we know that $\boldsymbol{x}(\overline{\tau_k}^+) \in \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$ and therefore there is a $t_3 \in (\overline{\tau_k}, t_1]$ such that

$$\begin{aligned} &\eta \le \eta_1 = \|\boldsymbol{x}(t_3) - \boldsymbol{x}_a(t_3)\| < \rho \text{ and} \\ &\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \rho \text{ for } t \in [t_0, t_3]. \end{aligned} \tag{5.105}$$

♠$_1$: Let us denote $w(\overline{\tau_i}^+) = V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}^+) - \boldsymbol{x}_a(\overline{\tau_i}^+))$ for $1 \le i \le k$ and suppose $\underline{\tau_i} = \tau_i^0$ such that

$$
\begin{aligned}
w(\overline{\tau_i}^+) &= V\Big(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i}))\\
&\quad + \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))]ds\Big)\\
&= V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i})))\\
&\quad + V\Big(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i}))\\
&\quad + \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))]ds\Big)\\
&\quad - V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i})))\\
&\le V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i})))\\
&\quad + L_2 \left\| \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))]ds \right\| \qquad \Leftarrow \text{(condition 3)}\\
&\le V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i}) + U(i, \boldsymbol{x}(\underline{\tau_i})) - U(i, \boldsymbol{x}_a(\overline{\tau_i})))\\
&\quad + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 1)}\\
&\le V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\overline{\tau_i})) + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 4)}\\
&= V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}))\\
&\quad + V\left(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) - \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}_a(s)) + \boldsymbol{u}(s, \boldsymbol{x}_a(s))]ds\right)\\
&\quad - V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i})\\
&\le V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}))\\
&\quad + L_2 \left\| - \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}_a(s)) + \boldsymbol{u}(s, \boldsymbol{x}_a(s))]ds \right\| \qquad \Leftarrow \text{(condition 3)}\\
&\quad + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i})\\
&\le V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) + 2L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 1)}\\
&\le w(\underline{\tau_i}) + 2L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}).
\end{aligned} \tag{5.106}
$$

Then it follows from condition 2 that $\tau_i(\boldsymbol{x}_a(\overline{\tau_i})) \le \tau_i(\boldsymbol{x}_a(\underline{\tau_i}))$ and

$$
\begin{aligned}
0 \le \overline{\tau_i} - \underline{\tau_i} &= \tau_i(\boldsymbol{x}_a(\overline{\tau_i})) - \tau_i(\boldsymbol{x}(\underline{\tau_i}))\\
&\le \tau_i(\boldsymbol{x}_a(\underline{\tau_i})) - \tau_i(\boldsymbol{x}(\underline{\tau_i}))\\
&\le K_i \|\boldsymbol{x}_a(\underline{\tau_i}) - \boldsymbol{x}(\underline{\tau_i})\| \qquad \Leftarrow \text{(condition 2)}\\
&\le \frac{K_i w(\underline{\tau_i})}{L_3}. \qquad \Leftarrow \text{(condition 6)}
\end{aligned} \tag{5.107}
$$

It then follows from (5.106) and (5.107) that

$$
w(\overline{\tau_i}^+) \le \left(1 + \frac{2L_1 L_2 K_i}{L_3}\right) w(\underline{\tau_i}). \tag{5.108}
$$

$\spadesuit_2$: Then let us suppose $\overline{\tau_i} = \tau_i^0$ such that

$$\begin{aligned}
w(\overline{\tau_i}^+) &= V\Big(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i})) \\
&\quad - \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}_a(s)) + \boldsymbol{u}(s, \boldsymbol{x}_a(s))] ds\Big) \\
&= V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i}))) \\
&\quad + V\Big(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i})) \\
&\quad - \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}_a(s)) + \boldsymbol{u}(s, \boldsymbol{x}_a(s))] ds\Big) \\
&\quad - V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i}))) \\
&\leq V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i}))) \\
&\quad + L_2 \left\| - \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}_a(s)) + \boldsymbol{u}(s, \boldsymbol{x}_a(s))] ds \right\| \qquad \Leftarrow \text{(condition 3)} \\
&\leq V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + U(i, \boldsymbol{x}(\overline{\tau_i})) - U(i, \boldsymbol{x}_a(\underline{\tau_i}))) \\
&\quad + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 1)} \\
&\leq V(\overline{\tau_i}, \boldsymbol{x}(\overline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 4)} \\
&= V\left(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))] ds\right) \\
&\quad + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \\
&= V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) \\
&\quad + V\left(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i}) + \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))] ds\right) \\
&\quad - V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \\
&\leq V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) \\
&\quad + L_2 \left\| \int_{\underline{\tau_i}}^{\overline{\tau_i}} [\boldsymbol{f}(s, \boldsymbol{x}(s)) + \boldsymbol{u}(s, \boldsymbol{x}(s))] ds \right\| \qquad \Leftarrow \text{(condition 3)} \\
&\quad + L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \\
&\leq V(\overline{\tau_i}, \boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})) + 2 L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}) \qquad \Leftarrow \text{(condition 1)} \\
&\leq w(\underline{\tau_i}) + 2 L_1 L_2 (\overline{\tau_i} - \underline{\tau_i}).
\end{aligned} \tag{5.109}$$

By using the same process leading to (5.107) we have

$$
\begin{aligned}
0 \le \overline{\tau_i} - \underline{\tau_i} &= \tau_i(\boldsymbol{x}(\overline{\tau_i})) - \tau_i(\boldsymbol{x}_a(\underline{\tau_i})) \\
&\le \tau_i(\boldsymbol{x}(\underline{\tau_i})) - \tau_i(\boldsymbol{x}_a(\underline{\tau_i})) \\
&\le K_i \|\boldsymbol{x}(\underline{\tau_i}) - \boldsymbol{x}_a(\underline{\tau_i})\| \qquad \Leftarrow \text{(condition 2)} \\
&\le \frac{K_i w(\underline{\tau_i})}{L_3} \qquad \Leftarrow \text{(condition 6)}
\end{aligned} \tag{5.110}
$$

which leads to the same estimate in (5.108). Then we have

$$
\begin{aligned}
D^+ w(t) &\le 0, \quad t \ne \overline{\tau_i}, \quad t \in [t_0, t_3], \\
w(\overline{\tau_i}^+) &\le \left(1 + \frac{2L_1L_2K_i}{L_3}\right) w(\underline{\tau_i})
\end{aligned} \tag{5.111}
$$

from which and Corollary 5.2.2 we have

$$
w(t) \le w(t_0^+) \prod_{i=1}^{k} \left(1 + \frac{2L_1L_2K_i}{L_3}\right), \quad t \in [t_0, t_3] \setminus \bigcup_{i=1}^{k} (\underline{\tau_i}, \overline{\tau_i}]. \tag{5.112}
$$

Then from (5.95), (5.96) and (5.105) we have the following contradiction:

$$
\begin{aligned}
&\underbrace{L_3\eta \le L_3\|\boldsymbol{x}(t_3)}_{(5.105)} \underbrace{-\boldsymbol{x}_a(t_3)\| \le w(}_{\text{(condition 6)}} \underbrace{t_3) \le w(t_0^+)}_{(5.112)} \prod_{i=1}^{k} \left(1 + \frac{2L_1L_2K_i}{L_3}\right) \\
&\le w(t_0^+) \underbrace{L_4 \le L_2L_4}_{(5.95)} \|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| < \underbrace{L_2L_4\varepsilon < L_2\eta}_{(5.96)}.
\end{aligned} \tag{5.113}
$$

Hence, Claim 5.3.1a is true and it follows from (5.96) and (5.107) we have

$$
\begin{aligned}
&\underbrace{\overline{\tau_i} - \underline{\tau_i} \le \frac{K_i w(\underline{\tau_i})}{L_3}}_{(5.107)} \le \frac{K_i w(\underline{\tau_i})}{L_3} \\
&\le \frac{K}{L_3} w(t_0^+) \prod_{k=1}^{i} \left(1 + \frac{2L_1L_2K_k}{L_3}\right) \\
&\le \frac{L_4K}{L_3} w(t_0^+) \le \frac{L_2L_4K}{L_3} \|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| \\
&< \underbrace{\frac{KL_2L_4\varepsilon}{L_3} < \frac{\xi}{2}}_{(5.96)}, \quad i \in \mathbb{N}
\end{aligned} \tag{5.114}
$$

from which it follows that whenever $\|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| < \varepsilon$, we have for all $i \in \mathbb{N}$, $t \ge t_0$ and $|t - \tau_i| > \xi$

$$
\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \eta.
$$

This proves that $\boldsymbol{x}_a(t)$ is uniformly stable. ∎

Theorem 5.3.2. *Let $\boldsymbol{x}_a(t) = \boldsymbol{x}_a(t, t_0, \boldsymbol{x}_{a0})$ be any given solution of system (5.93) on $[t_0, \infty)$ which hits the switching surfaces at moments τ_k^a, $k \in \mathbb{N}$, and let us suppose that the following conditions are satisfied:*

1. $\|\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})\| \le L_1$, $L_1 > 0$ *and* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$;
2. *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$ *and for all* k, *we have*
$$\frac{\partial \tau_k(\boldsymbol{x})}{\partial \boldsymbol{x}}[\boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x})] \le 0 \text{ and } |\tau_k(\boldsymbol{x}) - \tau_k(\boldsymbol{y})| \le K_k \|\boldsymbol{x} - \boldsymbol{y}\|$$
where $K_k \ge 0$ *and* $\sum_{k=1}^{\infty} K_k$ *converges;*
3. $V(t, \boldsymbol{x}) \in C[\mathbb{R}_+ \times \mathcal{S}_\rho, \mathbb{R}_+]$ *is nonincreasing in* t *for fixed* $\boldsymbol{x}$ *and there is an* $L_2 > 0$ *such that*
$$|V(t, \boldsymbol{x}) - V(t, \boldsymbol{y})| \le L_2 \|\boldsymbol{x} - \boldsymbol{y}\|$$
where $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$ *and* $(t, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$;
4. *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$, $(t, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$ *and* $k \in \mathbb{N}$
$$V(t, \boldsymbol{x} - \boldsymbol{y} + U(k, \boldsymbol{x}) - U(k, \boldsymbol{y})) \le V(t, \boldsymbol{x} - \boldsymbol{y});$$
5. *for* $\boldsymbol{x} \in \mathcal{S}_\rho$ *and for* $k \in \mathbb{N}$, $\|U(k, \boldsymbol{x})\| \le \rho/3$;
6. $V(t, 0) = 0$ *and there is an* $L_3 > 0$ *such that*
$$L_3 \|\boldsymbol{x} - \boldsymbol{x}_a(t)\| \le V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)), \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)};$$
7. $D^+V(t, \boldsymbol{x} - \boldsymbol{x}_a(t)) \le -\gamma(t)\zeta(\boldsymbol{x} - \boldsymbol{x}_a(t))$, $t \ne \tau_k(\boldsymbol{x})$, $t \ne \tau_k^a$, $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho^{\boldsymbol{x}_a(t)}$, *where* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is measurable and* $\zeta \in C[\mathcal{S}_\rho, \mathbb{R}_+]$;
8. *for any* $\delta > 0$ *there is a* $\delta_1 > 0$ *such that* $S \subset \mathbb{R}_+$ *and* $\mu(S) \ge \delta$ *implies* $\int_S \gamma(s) ds \ge \delta_1$, *where* μ *is Lebesque measure ;*
9. *for any* $\epsilon_1 > 0$ *there is an* $\epsilon = \epsilon(\epsilon_1) > 0$ *such that*
$$\epsilon_1 \le \|\boldsymbol{x}\| < \rho \Rightarrow \zeta(\boldsymbol{x}) \ge \epsilon.$$

Then $\boldsymbol{x}_a(t)$ *is uniformly asymptotically stable.* ⊠

Proof. From Theorem 5.3.1 and the proof of Theorem 5.3.1 we know that $\boldsymbol{x}_a(t)$ is uniformly stable and there is a $\varepsilon_0 > 0$ such that for any a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, of system (5.93) we have

$$\|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| < \varepsilon_0 \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \rho, \quad t \ge t_0 \tag{5.115}$$

To prove that $\boldsymbol{x}_a(t)$ is uniformly asymptotically stable, we have the following claim:
Claim 5.3.2a: For any give $\eta \in (0, \rho/3)$ and $\xi > 0$, there is a $T = T(\eta, \xi) > 0$ such that

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \eta, \quad t \ge t_0 + T, \quad |t - \tau_k^a| > \xi. \tag{5.116}$$

Since $\boldsymbol{x}_a(t)$ is uniformly stable, Claim 5.3.2a can be further simplified as the following claim:
Claim 5.3.2b: For any give $\eta \in (0, \rho/3)$ and $\xi > 0$, there is a $T = T(\eta, \xi) > 0$ such that for some

$$t_1 \in [t_0, t_0 + T] \setminus \bigcup_{\overline{\tau_k} \in (t_0, t_0+T)} [\underline{\tau_k}, \overline{\tau_k}]$$

we have

$$\|\boldsymbol{x}(t_1) - \boldsymbol{x}_a(t_1)\| < \varepsilon(\eta, \xi).$$

If Claim 5.3.2b is false, then for any $T > 0$ there is a solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (5.93) with $\|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| < \varepsilon_0$ such that

$$\varepsilon \leq \|\boldsymbol{x}(t) - \boldsymbol{x}_a(t)\| < \rho, \quad t \in [t_0, t_0 + T] \setminus \bigcup_{\overline{\tau_k} \in (t_0, t_0+T)} [\underline{\tau_k}, \overline{\tau_k}]. \quad (5.117)$$

From (5.94) we have

$$\begin{aligned} w(\underline{\tau_k}) &\leq L_2 \|\boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\underline{\tau_k})\| \\ &< L_2 \rho \qquad \Leftarrow (5.115) \end{aligned} \quad (5.118)$$

Because $\sum_{k=1}^{\infty} K_k$ is convergent, there is an $L_4 > 0$ such that $\sum_{k=1}^{\infty} K_k \leq L_4$. It then follows from (5.107) and (5.118) that

$$\begin{aligned} \sum_{k=1}^{\infty} (\overline{\tau_k} - \underline{\tau_k}) &\leq \sum_{k=1}^{\infty} \frac{K_k}{L_3} w(\underline{\tau_k}) \qquad \Leftarrow (5.107) \\ &\leq \frac{\rho L_2 L_4}{L_3} \qquad \Leftarrow (5.118) \end{aligned} \quad (5.119)$$

where $w(\underline{\tau_k}) = V(t, \boldsymbol{x}(\underline{\tau_k}) - \boldsymbol{x}_a(\underline{\tau_k}))$.

It follows from condition 9 that there is a $K = K(\varepsilon) > 0$ such that

$$\zeta(\boldsymbol{x}) \geq K \text{ for } \varepsilon \leq \|\boldsymbol{x}\| < \rho. \quad (5.120)$$

It follows from the assumption on $\gamma(t)$ (condition 8), we know that there is an $L_5 > 0$ such that $S \subset \mathbb{R}_+$ and $\mu(S) \geq L_5$ imply

$$\int_S \gamma(s) ds > \frac{L_2 \varepsilon_0 + 2 L_1 L_2 L_4 / L_3}{K}. \quad (5.121)$$

Let us set

$$T = L_5 + \frac{\rho L_2 L_4}{L_3} + 1$$

and $w(t) = V(t, \boldsymbol{x}(t) - \boldsymbol{x}_a(t))$, where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is the solution of system (5.93) satisfying (5.117). It follows the conclusion in (5.108) that

$$\sum_{k=1}^{\infty}[w(\overline{\tau_k}^+) - w(\underline{\tau_k})] \leq \sum_{k=1}^{\infty} \frac{2L_1L_2K_k}{L_3} \leq \frac{2L_1L_2L_4}{L_3}. \tag{5.122}$$

From (5.94) we have

$$\begin{aligned} w(t_0^+) &\leq L_2\|\boldsymbol{x}_0 - \boldsymbol{x}_{a0}\| \\ &< L_2\varepsilon_0 \qquad\qquad \Leftarrow (5.115) \end{aligned} \tag{5.123}$$

From (5.117), (5.119), (5.120), (5.121), (5.122) and condition 7 we have the following contradiction:

$$\begin{aligned} 0 &\leq w(t_0+T) \leq w(t_0^+) + \sum_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T} [w(\overline{\tau_k}^+) - w(\underline{\tau_k})] \\ &\quad + \int_{[t_0,t_0+T]\setminus\bigcup_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T}[\underline{\tau_k},\overline{\tau_k}]} D^+w(s)ds \\ &\leq w(t_0^+) + \sum_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T} [w(\overline{\tau_k}^+) - w(\underline{\tau_k})] \\ &\quad - \int_{[t_0,t_0+T]\setminus\bigcup_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T}[\underline{\tau_k},\overline{\tau_k}]} \gamma(s)\zeta(\boldsymbol{x}(s) - \boldsymbol{x}_a(s))ds \quad \Leftarrow \text{(condition 7)} \\ &< \underbrace{L_2\varepsilon_0}_{(5.123)} + \underbrace{\frac{2L_1L_2L_4}{L_3}}_{(5.122)} \underbrace{- K\int_{[t_0,t_0+T]\setminus\bigcup_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T}[\underline{\tau_k},\overline{\tau_k}]} \gamma(s)ds}_{\text{condition 7,(5.117)\&(5.120)}} \\ &< 0 \qquad \Leftarrow (5.121) \end{aligned} \tag{5.124}$$

because let

$$S = [t_0, t_0+T] \setminus \bigcup_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T} [\underline{\tau_k}, \overline{\tau_k}],$$

we have

$$\begin{aligned} \mu(S) &= T - \sum_{t_0<\underline{\tau_k}\leq\overline{\tau_k}<t_0+T} (\overline{\tau_k} - \underline{\tau_k}) \\ &\geq T - \sum_{k=1}^{\infty}(\overline{\tau_k} - \underline{\tau_k}) \\ &\geq T - \frac{L_2\rho}{L_3}\sum_{k=1}^{\infty} K_k \qquad \Leftarrow (5.119) \\ &\geq L_5 + \frac{L_2L_4\rho}{L_3} + 1 - \frac{L_2L_4\rho}{L_3} > L_5. \end{aligned} \tag{5.125}$$

Therefore Claim 5.3.2a is true and $\boldsymbol{x}_a(t)$ is uniformly asymptotically stable.

■

6. Practical Stability of Impulsive Control

In this chapter, we study the practical stability of impulsive control systems. The so called practical stability are very useful for designing practical controllers because in many cases, control a system to an idealized point is either expensive or impossible because of the finite measuring accuracy of sensors and actuators.

6.1 Practical Stability Based on Single Comparison System

Let us study the practical stability of the following impulsive control system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k, \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k \in \mathbb{N}. \end{aligned} \tag{6.1}$$

Definition 6.1.1. *The trivial solution of system (6.1) is*

- S1 *equi-stable if for each $\delta > 0$, $t_0 \in \mathbb{R}_+$, there is a $\xi = \xi(t_0, \delta)$ which is continuous in t_0 for each δ such that $\|\boldsymbol{x}_0\| < \xi$ implies $\|\boldsymbol{x}(t)\| < \delta$ for $t \geq t_0$;*
- S2 *uniformly stable if ξ in* S1 *is independent of t_0;*
- S3 *qusi-equi asymptotically stable, if for each $\delta > 0$, $t_0 \in \mathbb{R}_+$, there is a $\xi_0 = \xi_0(t_0) > 0$ and $T = T(t_0, \delta)$ such that $\|\boldsymbol{x}_0\| < \xi_0$ implies $\|\boldsymbol{x}(t)\| < \delta$ for $t \geq t_0 + T$;*
- S4 *qusi-uniformly asymptotically stable if ξ_0 and T in* S3 *are independent of t_0;*
- S5 *equi-asymptotically stable if both* S1 *and* S3 *hold;*
- S6 *uniformly asymptotically stable if both* S2 *and* S4 *hold;*
- S7 *qusi-equi asymptotically stable in the large, if for each $\delta > 0$, $\eta > 0$ $t_0 \in \mathbb{R}_+$, there is a $T = T(t_0, \delta, \eta) > 0$ such that $\|\boldsymbol{x}_0\| < \eta$ implies $\|\boldsymbol{x}(t)\| < \delta$ for $t \geq t_0 + T$;*

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 149−197, 2001.

S8 *qusi-uniformly asymptotically stable in the large if* T *in* S7 *is independent of* t_0*;*
S9 *completely stable if* S1 *and* S7 *hold for all* $\eta \in \mathbb{R}_+$*;*
SA *uniformly completely stable if* S2 *and* S8 *hold for all* $\eta \in \mathbb{R}_+$*;*
SB *unstable if* S1 *does not hold.*

⊠

Definition 6.1.2. *Given* (μ, ν) *with* $0 < \mu < \nu$ *and* $t_0 \in \mathbb{R}_+$*, and let* $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0), t \geq t_0$*, be a solution of system (6.1), then the impulsive control system (6.1) is said to be*

PS1 *practically stable if* $\|\boldsymbol{x}_0\| < \mu$ *implies* $\|\boldsymbol{x}(t)\| < \nu$ *for some* $t_0 \in \mathbb{R}_+$ *and* $t \geq t_0$*;*
PS2 *uniformly practically stable if* PS1 *holds for every* $t_0 \in \mathbb{R}_+$*;*
PS3 *practically quasistable if given* $\theta > 0$*,* $T > 0$ *and for some* $t_0 \in \mathbb{R}_+$*,* $\|\boldsymbol{x}_0\| < \mu$ *implies* $\|\boldsymbol{x}(t)\| < \theta$*,* $t \geq t_0 + T$*;*
PS4 *uniformly practically quasistable if* PS3 *holds for every* $t_0 \in \mathbb{R}_+$*;*
PS5 *strongly practically stable if both* PS1 *and* PS3 *hold;*
PS6 *strongly uniformly practically stable if both* PS2 *and* PS4 *hold;*
PS7 *practically asymptotically stable if both* PS1 *and* S7 *hold with* $\eta = \mu$*;*
PS8 *uniformly practically asymptotically stable if both* PS2 *and* S8 *hold with* $\eta = \mu$*;*
PS9 *practically unstable if* PS1 *does not hold;*
PSA *eventually practically stable if there is such a* $T = T(\mu, \nu)$ *that* $\|\boldsymbol{x}_0\| < \mu$ *implies* $\|\boldsymbol{x}(t)\| < \nu$*,* $t \geq t_0 \geq T$*;*
PSB *eventually strongly practically stable if both* PS3 *and* PSA *hold;*
PSC *eventually uniformly strongly practically stable if both* PS4 *and* PSA *hold.*

⊠

Let us define

$$\mathbb{D}^+_{\boldsymbol{x}}(\boldsymbol{y}) \triangleq \lim_{h \to 0^+} \frac{\|\boldsymbol{x} + h\boldsymbol{y}\| - \|\boldsymbol{x}\|}{h},$$

and

$$\mathbb{D}^-_{\boldsymbol{x}}(\boldsymbol{y}) \triangleq \lim_{h \to 0^-} \frac{\|\boldsymbol{x} + h\boldsymbol{y}\| - \|\boldsymbol{x}\|}{h}.$$

Assume that there is a $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ such that for

$$\mathbb{D}^+_{\boldsymbol{x}}(\boldsymbol{f}(t, \boldsymbol{x})) \leq g(t, \|\boldsymbol{x}\|), \quad t \neq \tau_k, \tag{6.2}$$

and assume that there are $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k \in \mathbb{N}$, such that $\psi_k(w)$ is nondecreasing in w and

$$\|\boldsymbol{x} + U(k, \boldsymbol{x})\| \leq \psi_k(\|\boldsymbol{x}\|), \tag{6.3}$$

then we have the following comparison system for the impulsive control system (6.1)

$$\begin{aligned} \dot{w} &= g(t, w), \quad t \neq \tau_k, \\ w(\tau_k^+) &= \psi_k(w(\tau_k)), \quad t = \tau_k, \\ w(t_0^+) &= w_0 \geq 0, \quad k \in \mathbb{N}. \end{aligned} \tag{6.4}$$

Theorem 6.1.1. *Let us assume that* $f \in C[(\tau_{k-1}, \tau_k] \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $\frac{\partial \boldsymbol{f}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}$ *is continuous in* $(\tau_{k-1}, \tau_k] \times \mathbb{R}^n$ *for each* $k \in \mathbb{N}$*, then the practical stability properties of the comparison system (6.4) imply those of the impulsive control system (6.1).* ⊠

Proof. Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be any a solution of control system (6.1), then we can set $\varpi(t) = \|\boldsymbol{x}(t)\|$, from the comparison system (6.4) it follows that

$$\begin{aligned} \dot{\varpi} &= g(t, \varpi), \quad t \neq \tau_k, \\ \varpi(\tau_k^+) &= \psi_k(\varpi(\tau_k)), \quad t = \tau_k, \\ \varpi(t_0^+) &= \|\boldsymbol{x}_0\|, \quad k \in \mathbb{N}. \end{aligned} \tag{6.5}$$

Let $w_{\max}(t) = w_{\max}(t, t_0, w_0)$ be the maximal solution of (6.4), then it follows from Theorem 3.1.1 that for $t \geq t_0$

$$\varpi(t) \leq w_{\max}(t, t_0, \|\boldsymbol{x}_0\|). \tag{6.6}$$

This finishes the proof. ■

Let us study the practical stability of the following linear impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x}, \quad t \neq \tau_k, \\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x}, \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k \in \mathbb{N}. \end{aligned} \qquad (6.7)}$$

Observe that $\mathbb{D}_{\boldsymbol{x}}^+(A\boldsymbol{x}) \leq \mu(A)\|\boldsymbol{x}\|$ where $\mu(A)$ is the logarithmic norm of A. Let us assume that $\|(I + B_k)\boldsymbol{x}\| \leq d_k\|\boldsymbol{x}\|$, then we have the following comparison system:

$$\begin{aligned} \dot{w} &= \mu(A)w, \quad t \neq \tau_k, \\ w(\tau_k^+) &= d_k w(\tau_k), \quad t = \tau_k, \\ w(t_0^+) &= w_0 \geq 0, \quad k \in \mathbb{N}, \end{aligned} \tag{6.8}$$

whose solution is given by

$$w(t, t_0, w_0) = w_0 \prod_{t_0 < \tau_k < t} d_k e^{\mu(A)(t-t_0)}, \quad t \geq t_0. \tag{6.9}$$

Then we have the following corollary.

Corollary 6.1.1. *Give $0 < \mu < \nu$ and assume that*

$$\prod_{k=1}^{\infty} d_k < \frac{\nu}{\mu}$$

and $\mu(A) \leq 0$ then the linear impulsive control system (6.7) is practically stable. ⊠

Proof. From the assumption we know that the comparison system (6.8) is practically stable. Therefore, it follows from Theorem 6.1.1 that the linear impulsive control system (6.7) is practically stable. ■

Since the usefulness of the practical stability of comparison systems, let us study the practical stability of the following special case of the comparison system (6.4)

$$\begin{aligned} \dot{w} &= g_1(t)g_2(w), \qquad t \neq \tau_k, \\ w(\tau_k^+) &= \psi_k(w(\tau_k)), \qquad t = \tau_k, \\ w(t_0^+) &= w_0 \geq 0, \qquad k \in \mathbb{N} \end{aligned} \tag{6.10}$$

where $g_1 \in C[\mathbb{R}_+, \mathbb{R}_+]$, $g_2 \in C[\mathbb{R}_+, \mathbb{R}_+]$ is nondecreasing and $\psi_k \in C[\mathbb{R}_+, \mathbb{R}_+]$, $k \in \mathbb{N}$ are also nondecreasing.

Theorem 6.1.2. *Assume that*

1. *given $0 < \mu < \nu$ such that*

$$\mu < \min_{k \in \mathbb{N}}(\nu, \psi_k(\nu));$$

2. *for every $\theta \geq \mu$ we have*

$$\int_{\tau_k}^{\tau_{k+1}} g_1(s)ds + \int_{\theta}^{\psi_k(\theta)} \frac{1}{g_2(s)} ds \leq 0, \quad k \in \mathbb{N}. \tag{6.11}$$

Then the comparison system (6.10) is practically stable. ⊠

Proof. Without loss of generality, let us assume that $t_0 \in (\tau_k, \tau_{k+1}]$ and let $w_0 \in [0, \mu)$, then we have the following claim:
Claim 6.1.2: $w(t) < \nu$ for all $t \geq t_0$.

If Claim 6.1.2 is false, then there is such a $t_1 \in (t_0, \tau_{k+1}]$ that $w(t_1) \geq \nu$. We then obtain

$$
\begin{aligned}
\int_{\psi_k(\nu)}^{\nu} \frac{1}{g_2(s)} ds &< \int_{\mu}^{\nu} \frac{1}{g_2(s)} ds \qquad \Leftarrow \text{(assumption 1)} \\
&\le \int_{w_0}^{\nu} \frac{1}{g_2(s)} ds \qquad \Leftarrow ([w_0 \in [0, \mu)] \\
&\le \int_{w_0}^{w(t_1)} \frac{1}{g_2(s)} ds \qquad \Leftarrow [w(t_1) \ge \nu] \\
&\le \int_{t_0}^{t_1} g_1(s) ds \qquad \Leftarrow [g_1(t) = \dot{w}/g_2(w)] \\
&\le \int_{\tau_k}^{\tau_{k+1}} g_1(s) ds \\
&\Uparrow \{t_0 \in (\tau_k, \tau_{k+1}] \;\&\; t_1 \in (t_0, \tau_{k+1}]\}
\end{aligned} \tag{6.12}
$$

from which we have the following contradiction to (6.11):

$$
\int_{\tau_k}^{\tau_{k+1}} g_1(s) ds + \int_{\nu}^{\psi_k(\nu)} \frac{1}{g_2(s)} ds > 0. \tag{6.13}
$$

Therefore, if $w_0 < \mu$ and $t \in (\tau_k, \tau_{k+1}]$ then $w(t) < \nu$.

We then use mathematical induction to prove that Claim 6.1.2 is true. Let us assume that $i \ge k+2$, $w(t) < \nu$ for $t \in (\tau_{k+1}, \tau_i]$, then for $t \in (\tau_i, \tau_{i+1}]$ we have

$$
\begin{aligned}
\int_{w(\tau_i^+)}^{w(t)} \frac{1}{g_2(s)} ds &\le \int_{\tau_i}^{t} g_1(s) ds \qquad \Leftarrow [g_1(t) = \dot{w}/g_2(w)] \\
&\le \int_{\tau_i}^{\tau_{i+1}} g_1(s) ds. \qquad \Leftarrow \{t \in (\tau_i, \tau_{i+1}]\}
\end{aligned} \tag{6.14}
$$

It follows from $w(\tau_i^+) = \psi_i(w(\tau_i))$ that

$$
\int_{w(\tau_i)}^{w(\tau_i^+)} \frac{1}{g_2(s)} ds = \int_{w(\tau_i)}^{\psi_i(w(\tau_i))} \frac{1}{g_2(s)} ds \tag{6.15}
$$

from which and (6.14) we have

$$
\begin{aligned}
\int_{w(\tau_i)}^{w(t)} \frac{1}{g_2(s)} ds &= \int_{w(\tau_i)}^{w(\tau_i^+)} \frac{1}{g_2(s)} ds + \int_{w(\tau_i^+)}^{w(t)} \frac{1}{g_2(s)} ds \\
&\le \int_{w(\tau_i)}^{\psi_i(w(\tau_i))} \frac{1}{g_2(s)} ds + \int_{\tau_i}^{\tau_{i+1}} g_1(s) ds \le 0.
\end{aligned} \tag{6.16}
$$

Therefore we have $w(t) \le w(\tau_i) < \nu$ for $t \in (\tau_k, \tau_{i+1}]$. By using mathematical induction we know that Claim 6.1.2 is true. This proves that the comparison system (6.10) is practically stable. ■

Let us study the practical stability of the following nonlinear impulsive control system by using first-order approximation

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + \boldsymbol{u}(t, \boldsymbol{x}), \quad t \neq \tau_k, \\ \Delta \boldsymbol{x} &= B_k \boldsymbol{x} + \boldsymbol{c}_k(\boldsymbol{x}), \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k \in \mathbb{N}. \end{aligned} \qquad (6.17)}$$

where $A \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^{n\times n}]$, $\boldsymbol{f} \in C[(\tau_{k-1}, \tau_k] \times \mathbb{R}^n$ and for every $\boldsymbol{x} \in \mathbb{R}^n$, $k \in \mathbb{N}$ the following limit exists:

$$\lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \boldsymbol{f}(t, \boldsymbol{y}) = \boldsymbol{f}(\tau_k^+, \boldsymbol{x}),$$

$B_k \in \mathbb{R}^{n\times n}$ and $\boldsymbol{c}_k \in C[\mathbb{R}^n, \mathbb{R}^n]$ for $k \in \mathbb{N}$.

Let $\Phi_k(t, s)$ be the fundamental matrix solution of the following reference system

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}, \quad t \in (\tau_{k-1}, \tau_k]. \tag{6.18}$$

Let us assume that

HA $\|\boldsymbol{u}(t, \boldsymbol{x})\| \leq \alpha(t)\|\boldsymbol{x}\|^\sigma$, $\sigma > 1$, for $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\alpha \in \mathcal{PC}[\mathbb{R}_+, (0, \infty)]$.
HB $\|I + B_k\| \leq \kappa_k$, $\kappa_k \geq 0$, $\|\boldsymbol{c}_k(\boldsymbol{x})\| \leq \gamma_k\|\boldsymbol{x}\|$, $\gamma_k \geq 0$, for $k \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$.
HC $\|\Phi_k(t, s)\| \leq g_1(t)g_2(s)$ for $\tau_{k-1} < s \leq t \leq \tau_k$, $g_1(\tau_k) > 0$, $g_2(\tau_k^+) > 0$ for $k \in \mathbb{N}$, $g_1 \in \mathcal{PC}[\mathbb{R}_+, (0, \infty)]$, $g_2 \in \mathcal{PC}[\mathbb{R}_+, (0, \infty)]$.

Theorem 6.1.3. *Let us define* $\varpi_k \triangleq (\gamma_k + \kappa_k)g_1(\tau_k)g_2(\tau_k^+)$ *and*

$$\phi(t_0, t) \triangleq \int_{t_0}^{t} \left(\prod_{t_0<\tau_k<s} \varpi_k \right)^{\sigma-1} \alpha(s)g_1^\sigma(s)g_2(s)ds.$$

Assume that $\phi(t_0, \infty) < \infty$ *then the impulsive control system (6.17) is*

1. *practically stable if for a given* $K_1 > 0$

$$g_1(t)g_2(t_0^+) \prod_{t_0<\tau_k<t} \varpi_k < K_1, \quad t \geq t_0; \tag{6.19}$$

2. *strongly practically stable if for a given* $K_2 > 0$ *and* $T > 0$

$$g_1(t)g_2(t_0^+) \prod_{t_0<\tau_k<t} \varpi_k < K_2, \quad t \geq t_0 + T. \tag{6.20}$$

⊠

Proof. Let $\Psi(t, s)$ be the Cauchy matrix of the following reference system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= A(t)\boldsymbol{x}, \quad t \neq \tau_k,\\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x}, \quad t = \tau_k,\\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k \in \mathbb{N}.\end{aligned} \tag{6.21}$$

Then the solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (6.17) can be represented as

$$\begin{aligned}\boldsymbol{x}(t) = \Phi(t, t_0)\boldsymbol{x}_0 &+ \int_{t_0}^{t} \Psi(t, s)\boldsymbol{u}(s, \boldsymbol{x}(s))ds\\ &+ \sum_{t_0<\tau_k<t} \Psi(t, \tau_k^+)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)).\end{aligned} \tag{6.22}$$

Let us define

$$w(t) = \frac{\|\boldsymbol{x}(t)\|}{g_1(t)}$$

then it follows from (1.32), assumptions HA, HB and HC that

$$\begin{aligned}w(t) \le \|\boldsymbol{x}_0\| g_2(t_0^+) &\prod_{t_0<\tau_k<t} \kappa_k g_1(\tau_k) g_2(\tau_k^+)\\ &+ \int_{t_0}^{t}\left(\prod_{s\le\tau_k<t} \kappa_k g_1(\tau_k) g_2(\tau_k^+)\right)\alpha(s) g_1^{\sigma}(s) g_2(s) w^{\sigma}(s)ds\\ &+ \sum_{t_0<\tau_k<t} \gamma_k g_1(\tau_k) g_2(\tau_k^+) \prod_{\tau_k<\tau_i<t} \kappa_i g_1(\tau_i) g_2(\tau_i^+) w(\tau_k),\\ &t \ge t_0.\end{aligned} \tag{6.23}$$

From which, Theorem A.3.2 of [15] and Lemma A.1.1 of [15] it follows that

$$w(t) \le \|\boldsymbol{x}_0\| g_2(t_0^+) \prod_{t_0\tau_k<t} \varpi_k \{1 - (\sigma-1)[g_2(t_0^+)\|\boldsymbol{x}_0\|]^{\sigma-1}\phi(t_0, t)\}^{1/(1-\sigma)}, \tag{6.24}$$

for all $t \ge t_0$ such that $(\sigma-1)[g_2(t_0^+)\|\boldsymbol{x}_0\|]^{\sigma-1}\phi(t_0, t) < 1$.

Assume that $\phi(t_0, \infty) < \infty$ and (6.19) holds, then let us choose $\|\boldsymbol{x}_0\| < \mu$ and

$$\mu = \min\left([2(\sigma-1)g_2^{\sigma-1}(t_0^+)\phi(t_0, \infty)]^{1/(1-\sigma)}, \frac{\nu}{K_1}2^{1/(1-\sigma)}\right)$$

then it follows from (6.24) that $\|\boldsymbol{x}(t)\| < \nu$ for all $t \ge t_0$. Therefore, the impulsive control system (6.17) is practically stable .

Since (6.20) shows practical quasi-stability, the second conclusion can be easily proved. ■

Let us using comparison method to study the practical stability of the following nonlinear impulsive control system:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k, \\
\Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k, \\
\boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k \in \mathbb{N}
\end{aligned} \tag{6.25}
$$

where $\boldsymbol{f} \in C[(\tau_{k-1}, \tau_k] \times \mathbb{R}^n, \mathbb{R}^n]$ and for any $\boldsymbol{x} \in \mathbb{R}^n$, $k \in \mathbb{N}$, the following limit exists:

$$\lim_{(t,\boldsymbol{y}) \to (\tau_k^+, \boldsymbol{x})} \boldsymbol{f}(t, \boldsymbol{y}) = \boldsymbol{f}(\tau_k^+, \boldsymbol{x}).$$

$U : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ and

$$0 < \tau_1 < \tau_2 < \cdots \tau_k < \tau_{k+1} < \cdots, \quad \lim_{k \to \infty} \tau_k = \infty.$$

Assume that there is a $V \in \mathcal{V}_0$ such that

$$
\begin{aligned}
&D^+ V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x})), \quad t \neq \tau_k, \\
&V(t, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x})), \quad t = \tau_k, \quad k \in \mathbb{N},
\end{aligned} \tag{6.26}
$$

where $g \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k \in \mathbb{N}$, are nondecreasing. Then we have the following comparison system for system (6.25):

$$
\begin{aligned}
&\dot{w} = g(t, w), \quad t \neq \tau_k, \\
&w(\tau_k^+) = \psi_k(w(\tau_k)), \quad t = \tau_k, \quad k \in \mathbb{N}.
\end{aligned} \tag{6.27}
$$

Theorem 6.1.4. *Given $0 < \mu < \nu$, let us assume that*

1. *$V \in \mathcal{V}_0$ satisfies (6.26) for $\boldsymbol{x} \in \mathcal{S}_\nu$;*
2. *there is such a $\rho = \rho(\nu) > 0$ that for any $\boldsymbol{x} \in \mathcal{S}_\nu$ and $k \in \mathbb{N}$ we have $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_\rho$;*
3. *there are $\alpha, \beta \in \mathcal{K}$ such that $\alpha(\mu) < \beta(\nu)$ and for any $t \in \mathbb{R}_+$ and $\boldsymbol{x} \in \mathcal{S}_\rho$ we have*
$$\beta(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|).$$

Then the practical stability properties of the comparison system (6.27) imply those of the impulsive control system (6.25). ⊠

Proof.

1. Practical stability. Let us suppose that the comparison system (6.27) is practically stable; namely, for any solution $w(t) = w(t, t_0, w_0)$ of (6.27) we know that $w_0 < \alpha(\mu)$ implies
$$w(t) < \beta(\nu) \text{ for } t \geq t_0. \tag{6.28}$$
Let us choose $w_0 = \alpha(\|\boldsymbol{x}_0\|)$ and let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be any a solution of system (6.25) then we have the following claim:

Claim 6.1.4: $\|\boldsymbol{x}_0\| < \mu$ implies $\|\boldsymbol{x}(t)\| < \nu$ for all $t \geq t_0$.
If Claim 6.1.4 is false, then there is a $t_1 > t_0$ such that $t_1 \in (\tau_k, \tau_{k+1}]$ for some k, $\|\boldsymbol{x}_0\| < \mu$, $\|\boldsymbol{x}(t)\| < \nu$ for $t \in [t_0, \tau_k]$ and $\|\boldsymbol{x}(t_1)\| \geq \nu$. It follows from condition 2 that if $\|\boldsymbol{x}(\tau_k)\| < \nu$ then $\|\boldsymbol{x}(\tau_k^+)\| = \|\boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))\| < \rho$. Thus, there is a $t_2 \in (\tau_k, t_1]$ such that $\|\boldsymbol{x}(t_2)\| \in [\nu, \rho)$. Let us set $w(t) = V(t, \boldsymbol{x}(t))$ for $t \in [t_0, t_2]$ and in view of Theorem 3.1.2, conditions 1 and 2, we have

$$V(t, \boldsymbol{x}(t)) \leq w_{\max}(t, t_0, \alpha(\|\boldsymbol{x}_0\|)) \tag{6.29}$$

where $w_{\max}(t, t_0, w_0)$ is the maximal solution of (6.27). It follows from condition 3 and (6.29) that

$$\begin{aligned} \beta(\nu) &\leq \beta(\|\boldsymbol{x}(t_2)\|) \quad \Leftarrow [\|\boldsymbol{x}(t_2)\| \in [\nu, \rho)] \\ &\leq V(t_2, \boldsymbol{x}(t_2)) \quad \Leftarrow (\text{condition } 3) \\ &\leq w_{\max}(t_2, t_0, \alpha(\|\boldsymbol{x}_0\|)) \quad \Leftarrow (6.29) \\ &< \beta(\nu) \quad \Leftarrow (6.28) \end{aligned} \tag{6.30}$$

which leads to a contradiction. Therefore, Claim 6.1.4 is true and the system (6.25) is practically stable.

2. Strongly practical stability . Let us suppose that the comparison system (6.27) is strongly practically stable, then form the first conclusion of this theorem we know that system (6.25) is practically stable; namely,

$$\|\boldsymbol{x}_0\| < \mu \text{ implies } \|\boldsymbol{x}(t)\| < \nu \text{ for } t \geq t_0. \tag{6.31}$$

Given $0 < \theta < \nu$ and $T > 0$, then from the fact that (6.27) is practically quasi-stable we have

$$w_0 \in [0, \alpha(\mu)) \text{ implies } w(t, t_0, w_0) < \beta(\theta) \text{ for } t \geq t_0 + T. \tag{6.32}$$

Given $\|\boldsymbol{x}_0\| < \mu$, it follows from (6.31) and by using the same procedure that leading to (6.29) we have

$$V(t, \boldsymbol{x}(t)) \leq w_{\max}(t, t_0, \alpha(\|\boldsymbol{x}_0\|)), \quad t \geq t_0 \tag{6.33}$$

which yields for $t \geq t_0 + T$

$$\beta(\|\boldsymbol{x}(t)\|) \leq Vt, \boldsymbol{x}(t)) \leq w_{\max}(t, t_0, \alpha(\|\boldsymbol{x}_0\|)) < \beta(\theta). \tag{6.34}$$

This proves that $\|\boldsymbol{x}(t)\| < \theta$ for $t \geq t_0 + T$. Therefore, the impulsive control system (6.25) is strongly practically stable.

∎

We then the following corollary of Theorem 6.1.4.

Corollary 6.1.2. *We have the following conclusions:*

1. *Assume that* $g(t,w)=0$ *and* $\psi_k(w)=d_k w$ *where* $d_k \geq 0$, $k \in \mathbb{N}$ *and*

$$\prod_{k=1}^{\infty} d_k < \infty$$

then the impulsive control system (6.25) is uniformly practically stable.

2. *Assume that* $g(t,w)=\dot{\gamma}(t)w$, $\gamma \in C^1[\mathbb{R}_+,\mathbb{R}_+]$ *and* $\psi_k(w)=d_k w$ *where* $d_k \geq 0$, $k \in \mathbb{N}$ *and*

$$\gamma(\tau_k)+\ln d_k \leq \gamma(\tau_{k-1}) \tag{6.35}$$

for all $k \in \mathbb{N}$, *then the impulsive control system (6.25) is practically stable.*

3. *Assume that* $g(t,w)=\dot{\gamma}(t)w$, $\gamma \in C^1[\mathbb{R}_+,\mathbb{R}_+]$ *and* $\psi_k(w)=d_k w$ *where* $d_k \geq 0$, $k \in \mathbb{N}$ *and there is a* $\varpi > 1$ *such that*

$$\gamma(\tau_k)+\ln(\varpi d_k) \leq \gamma(\tau_{k-1}) \tag{6.36}$$

for all $k \in \mathbb{N}$, *then the impulsive control system (6.25) is practically asymptotically stable.*

⊠

Proof.

1. Let us prove conclusion 1 first. In this case let us suppose that there is a $K>1$ such that

$$\prod_{k=1}^{\infty} d_k < K.$$

For any ν let us choose $\mu=\nu/K$, then we have

$$w(\infty)=w(t_0)\prod_{t_0\leq\tau_k<\infty} d_k.$$

From which we know that for any $w_0<\mu$ we have $w(\infty)<\nu$. Therefore, system (6.27) is uniformly practically stable. Then it follows from Theorem 6.1.4 that conclusion 1 is valid.

2. We then prove conclusion 2. In this case, the solution of system (6.27) is given by

$$w(t,t_0,w_0)=w_0\left(\prod_{t_0\leq\tau_k<t} d_k\right)e^{\gamma(t)-\gamma(t_0)},\quad t\geq t_0.$$

Without loss of generality, let us assume that $t_0\in(0,\tau_1)$, then it follows from (6.35) that

$$w(t,t_0,w_0)\leq w_0 e^{\gamma(\tau_1)-\gamma(t_0)},\quad t\geq t_0.$$

Let us choose μ and ν such that

$$e^{\gamma(\tau_1)-\gamma(t_0)} < \frac{\beta(\nu)}{\alpha(\mu)}$$

then we prove that system (6.27) is practically stable. Then it follows from Theorem 6.1.4 that conclusion 2 is valid.

3. We then prove conclusion 3. In this case, by using similar process we can get

$$w(t, t_0, w_0) \leq \frac{w_0 e^{\gamma(\tau_1)-\gamma(t_0)}}{\varpi^k}, \quad \tau_{k-1} < t \leq \tau_k,$$

which yields

$$\lim_{t\to\infty} w(t, t_0, w_0) = 0.$$

Therefore, system (6.27) is practically asymptotically stable . Then it follows from Theorem 6.1.4 that conclusion 3 is valid.

■

Note 6.1.1. The results in this section are adopted from [15] with revisions.

♦

6.2 Practical Stability in Terms of Two Measures

In this section we study the practical stability in terms of two measures of the following general impulsive control system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad & t \neq \tau_k, \\ \Delta \boldsymbol{x} &= \boldsymbol{u}_k(\boldsymbol{x}), \quad & t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad & k = 1, 2, \cdots . \end{aligned} \tag{6.37}$$

Let Ω be the admissible control input for $\tilde{\boldsymbol{u}}$. Given a $\tilde{\boldsymbol{u}}_1 \in \Omega$, we denote by $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}_1)$ a solution of system (6.37) with $\boldsymbol{x}(t_0^+) = \boldsymbol{x}_0$.

6.2.1 Definitions and Notations

Definition 6.2.1. *Let $h_0, h \in \mathcal{H}$. Then the system (6.37) is said to be*

1. *(h_0, h)-practically stable if, given (λ, α) with $0 < \lambda < \alpha$, we have $h_0(t_0, \boldsymbol{x}_0) < \lambda$ implies $h(t, \boldsymbol{x}(t)) < \alpha$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$;*
2. *(h_0, h)-uniformly practically stable if (1) holds for every $t_0 \in \mathbb{R}_+$;*

3. *(h_0, h)-practically quasistable if, given (λ, α, T) with $\lambda > 0$, $\alpha > 0$, $T > 0$ and some $t_0 \in \mathbb{R}_+$, we have $h_0(t_0, \boldsymbol{x}_0) < \lambda$ implies $h(t, \boldsymbol{x}(t)) < \alpha$, $t \geq t_0 + T$;*
4. *(h_0, h)-uniformly practically quasistable if (3) holds for every $t_0 \in \mathbb{R}_+$;*
5. *(h_0, h)-strongly practically stable if both (1) and (3) hold;*
6. *(h_0, h)-strongly uniformly practically stable if both (2) and (4) hold;*
7. *(h_0, h)-practically unstable if (1) does not hold.*

⊠

Definition 6.2.1 is in terms of two measures which is a more general form than that in terms of one measure. Definition 6.2.1 reduces to

1. the standard practical stability if $h(t, \boldsymbol{x}(t)) = h_0(t, \boldsymbol{x}(t)) = \|\boldsymbol{x}\|$;
2. the practical stability of a prescribed solution $\boldsymbol{x}_e$ of system (6.37) if $h(t, \boldsymbol{x}(t)) = h_0(t, \boldsymbol{x}(t)) = \|\boldsymbol{x} - \boldsymbol{x}_e\|$;
3. the partial practical stability if $h(t, \boldsymbol{x}(t)) = \|\boldsymbol{x}_s\|$, $1 \leq s < n$, where $\boldsymbol{x}_s^\top = (x^{(1)}, x^{(2)}, \cdots, x^{(s)})$ is an s-vector whose entries are chosen from $\boldsymbol{x}$ and $h_0(t, \boldsymbol{x}(t)) = \|\boldsymbol{x}\|$;
4. the orbital practical stability of an orbit $\wp$ if $h(t, \boldsymbol{x}(t)) = h_0(t, \boldsymbol{x}(t)) = d(\boldsymbol{x}, \wp)$, where $d(\cdot, \cdot)$ is a distance function;
5. the practical stability of an invariant set $\wp$ if $h(t, \boldsymbol{x}(t)) = h_0(t, \boldsymbol{x}(t)) = d(\boldsymbol{x}, \wp)$, where $d(\cdot, \cdot)$ is a distance function;
6. the practical stability of conditionally invariant set $\wp_1$ with respect to $\wp_2$, where $\wp_2 \subset \wp_1$, if $h(t, \boldsymbol{x}(t)) = d(\boldsymbol{x}, \wp_1)$ and $h_0(t, \boldsymbol{x}(t)) = d(\boldsymbol{x}, \wp_2)$, where $d(\cdot, \cdot)$ is a distance function.

6.2.2 Comparison System

Let a *comparison system* be

$$\begin{cases} \dot{w} = g(t, w, w), & t \neq \tau_k, \\ w(\tau_k^+) = \psi_k(w(\tau_k)), & k = 1, 2, \cdots, \\ w(t_0^+) = w_0 \geq 0, \end{cases} \tag{6.38}$$

where $g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in $(\tau_{k-1}, \tau_k] \times \mathbb{R} \times \mathbb{R}$ and for each $(w, y) \in \mathbb{R} \times \mathbb{R}$, $k = 1, 2, \cdots$,

$$\lim_{(t, \tilde{w}, \tilde{y}) \to (\tau_{k-1}^+, w, y)} g(t, \tilde{w}, \tilde{y}) = g(\tau_{k-1}^+, w, y)$$

exists, and $\psi_k : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $k = 1, 2, \cdots$.

Lemma 6.2.1. *Let us suppose that*

1. $T > 0$, $m, v \in C[[t_1, t_1 + T], \mathbb{R}]$ *and*

$$D^+ m(t) \leq g(t, m(t), v(t)), \quad t \in [t_1, t_1 + T]$$

where $g \in C[[t_1, t_1 + T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is nondecreasing in v for every $(t, m) \in [t_1, t_1 + T] \times \mathbb{R}$;

2. *$w_{\max}(t)$ is the maximal solution on $[t_1, t_1 + T]$ of the following scalar ordinary differential equation*

$$\dot{w} = g(t, w, w), \quad w(t_1) = w_1 \geq 0 \tag{6.39}$$

and $v(t) \leq w_{\max}(t)$ for $t \in [t_1, t_1 + T]$.

Then for $t \in [t_1, t_1 + T]$, $m(t_1) \leq w_1$ implies $m(t) \leq w_{\max}(t)$. ⊠

Note 6.2.1. The proof of this theorem can be found in [14]. ♦

Theorem 6.2.1. *Assume that*

1. *$m, v \in \mathcal{PC}$ and*

$$\begin{aligned} D^+ m(t) &\leq g(t, m(t), v(t)), & t &\neq \tau_k, \\ m(\tau_k^+) &\leq \psi_k(m(\tau_k)), & k &= 1, 2, \cdots ; \end{aligned} \tag{6.40}$$

2. *$w_{\max}(t, t_0, w_0)$ is the maximal solution of (6.38) on $[t_0, \infty)$.*

Then $m(t) \leq w_{\max}(t, t_0, w_0)$ for $t \geq t_0$ if $m(t_0) \leq w_0$. ⊠

Proof. Let $w_{\max}(t) = w_{\max}(t, t_0, w_0)$ be the maximal solution of the comparison system (6.38), then by classical comparison theorem, it is easy to see that $w_{\max}(t)$ is the maximal solution of (6.39) with $t \in [\tau_{k-1}, \tau_k]$ and $w(\tau_{k-1}^+) = w_{\max}(\tau_{k-1}^+)$, $k \in \mathbb{N}$. Then it follows from Lemma 6.2.1 that for $t \in (t_0, \tau_1]$ we have

$$m(t) \leq w_{\max 1}(t, t_0, w_0) \tag{6.41}$$

where $w_{\max 1}(t, t_0, w_0)$ is the maximal solution of (6.39) for $t \in [t_0, \tau_1]$ with $w_{\max 1}(t_0^+, t_0, w_0) = w_0$. Let $w_1^+ = \psi_1(w_{\max 1}(\tau_1, t_0, w_0))$, since ψ_1 is nondecreasing, it follows (6.41) and assumption 1 that $m(\tau_1^+) \leq w_1^+$. Then by Lemma 6.2.1, we have

$$m(t) \leq w_{\max 2}(t, \tau_1, w_1^+), \quad t \in (\tau_1, \tau_2] \tag{6.42}$$

where $w_{\max 2}(t, \tau_1, w_1^+)$ is the maximal solution of (6.39) for $t \in [\tau_1, \tau_2]$ with $w_{\max 2}(\tau_1^+, \tau_1, w_1^+) = w_1^+$. Then by repeating the same process we have, for $k = 1, 2, \cdots$,

$$m(t) \leq w_{\max k}(t, \tau_{k-1}, w_{k-1}^+), \quad t \in (\tau_{k-1}, \tau_k] \tag{6.43}$$

where $w_{\max k}(t, \tau_{k-1}, w_{k-1}^+)$ is the maximal solution of (6.39) for $t \in [\tau_{k-1}, \tau_k]$ with $w_{\max k}(\tau_{k-1}^+, \tau_{k-1}, w_{k-1}^+) = w_{k-1}^+$. Let us construct the following solution of (6.38):

$$w(t) = \begin{cases} w_0, & t = t_0 \\ w_{\max 1}(t, t_0, w_0), & t \in (t_0, \tau_1], \\ w_{\max 2}(t, \tau_1, w_1^+), & t \in (\tau_1, \tau_2], \\ \vdots \\ w_{\max k}(t, \tau_{k-1}, w_{k-1}^+), & t \in (\tau_{k-1}, \tau_k], \\ \vdots \end{cases} \tag{6.44}$$

which satisfies $w(t) \leq w_{\max}(t, t_0, w_0)$. Since

$$m(t) \leq w(t), \quad t \geq t_0$$

we have

$$m(t) \leq w_{\max}(t, t_0, w_0), \quad t \geq t_0.$$

■

It follows from Theorem 6.2.1 we have the following lemma (Theorem 3.1 of [19], Theorem 3.3.1 of [11]) for later use.

Lemma 6.2.2. *Assume that*

1. $m, v \in \mathcal{PC}$ *and*

$$\begin{aligned} D^+ m(t) &\leq g(t, m(t), v(t)), \qquad t \neq \tau_k, \\ m(\tau_k^+) &\leq \psi_k(m(\tau_k)), \qquad k = 1, 2, \cdots; \end{aligned} \tag{6.45}$$

2. $w_{\max}(t, t_0, w_0)$ *is the maximal solution of (6.38) on* $[t_0, \infty)$ *such that* $v(t) \leq w_{\max}(t, t_0, w_0)$, $t \geq t_0$;
3. $g(t, w, v)$ *is nondecreasing in* v *for each* (t, w).

Then $m(t) \leq w_{\max}(t, t_0, w_0)$ *for* $t \geq t_0$ *if* $m(t_0) \leq w_0$. ⊠

For the purpose of designing impulsive controller, some special cases of Lemma 6.2.2 were found useful and are listed as follows.

Corollary 6.2.1. *If in Lemma 6.2.2, we choose*

1. $g(t, w, w) \equiv 0$ *and* $\psi_k(w) = w$, $k \in \mathbb{N}$, *then* $m(t) \leq w_0$ *for* $t \geq t_0$;
2. $g(t, w, w) \equiv 0$ *and* $\psi_k(w) = d_k w, d_k \geq 0$, $k \in \mathbb{N}$, *then*

$$m(t) \leq w_0 \prod_{t_0 < \tau_k < t} d_k \tag{6.46}$$

for $t \geq t_0$;

3. $g(t, w, w) = -\gamma w, \gamma > 0$ *and* $\psi_k(w) = d_k w, d_k \geq 0$, $k \in \mathbb{N}$, *then*

$$m(t) \leq w_0 e^{-\gamma(t - t_0)} \prod_{t_0 < \tau_k < t} d_k \tag{6.47}$$

for $t \geq t_0$;

4. *$\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$, $\dot{\lambda}(t) \geq 0$ such that $g(t,w,w) = \dot{\lambda}w, \gamma > 0$, and $\psi_k(w) = d_k w, d_k \geq 0$, $k \in \mathbb{N}$, then*

$$m(t) \leq w_0 e^{\lambda(t)-\lambda(t_0)} \prod_{t_0<\tau_k<t} d_k \tag{6.48}$$

for $t \geq t_0$;

5. *$g(t,w,w) = -\gamma w + \varrho, \gamma > 0, \varrho > 0$ and $\psi_k(w) = d_k w, d_k \geq 0$, $k \in \mathbb{N}$, then*

$$\begin{aligned} m(t) \leq w_0 e^{-\gamma(t-t_0)} \prod_{i=1}^{k} d_i + \frac{\varrho}{\gamma} \sum_{i=1}^{k} \prod_{j=i}^{k} d_j \left(e^{-\gamma(t-\tau_i)} - e^{-\gamma(t-\tau_{i-1})}\right) \\ + \frac{\varrho}{\gamma}\left(w - e^{-\gamma(t-\tau_k)}\right) \end{aligned} \tag{6.49}$$

for $t \in (\tau_k, \tau_{k+1}]$.

⊠

We then have the following comparison theorem.

Theorem 6.2.2. *Let us assume that*

1. *$0 < \mu < \nu$ are given and $\rho > \nu$;*
2. *$h_0, h \in \mathcal{H}$ and $h(t,\boldsymbol{x}) \leq \phi(h_0(t,\boldsymbol{x}))$ with $\phi \in \mathcal{K}$ whenever $h_0(t,\boldsymbol{x}) < \mu$;*
3. *$V \in \mathcal{V}_0$ and there are $\alpha, \beta \in \mathcal{K}$ such that*

$$\begin{aligned} \beta(h(t,\boldsymbol{x})) \leq V(t,\boldsymbol{x}), \qquad \textit{if} \quad h(t,\boldsymbol{x}) < \rho, \\ V(t,\boldsymbol{x}) \leq \alpha(h_0(t,\boldsymbol{x})), \qquad \textit{if} \quad h_0(t,\boldsymbol{x}) < \mu; \end{aligned} \tag{6.50}$$

4. *let*

$$\Omega = \{\tilde{\boldsymbol{u}} \in \mathbb{R}^m | \;\; U(t,\tilde{\boldsymbol{u}}) \leq \gamma(t), t \geq t_0\} \tag{6.51}$$

where $U : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ is continuous on $(\tau_{k-1}, \tau_k] \times \mathbb{R}^m$ and for every $\tilde{\boldsymbol{u}} \in \mathbb{R}^m, k = 1, 2, \cdots,$

$$\lim_{(t,\boldsymbol{y}) \to (t_{k-1}^+, \tilde{\boldsymbol{u}})} U(t,\boldsymbol{y}) = U(t_{k-1}^+, \tilde{\boldsymbol{u}})$$

exists, and $w_{\max}(t)$ is the maximal solution of (6.38). For $(t,\boldsymbol{x}) \in (\tau_{k-1}, \tau_k) \times \mathbb{R}^n$ and $\tilde{\boldsymbol{u}}(t) \in \Omega$,

$$D^+V(t,\boldsymbol{x}) \leq g(t, V(t,\boldsymbol{x}), U(t,\tilde{\boldsymbol{u}}(t))), \qquad \textit{if} \quad h(t,\boldsymbol{x}) < \rho, \tag{6.52}$$

where $g(t,w,v)$ is nondecreasing in v for each $(t,w) \in \mathbb{R}_+ \times \mathbb{R}$ and

$$V(\tau_k^+, \boldsymbol{x}(\tau_k^+)) \leq \psi_k(V(\tau_k, \boldsymbol{x}(\tau_k))), \qquad \textit{if} \quad h(\tau_k, \boldsymbol{x}(\tau_k)) < \rho; \tag{6.53}$$

5. *$\phi(\mu) < \nu$ and $\alpha(\mu) < \beta(\nu)$;*
6. *$h(t,\boldsymbol{x}) < \nu$ implies $h(t,\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x})) < \rho$ for all $k = 1, 2, \cdots$.*

Then the practical stability properties of (6.38) with respect to $(\alpha(\mu), \beta(\nu))$ *imply the corresponding* (h_0, h)*-practical stability properties of the impulsive control (6.37) with respect to* (μ, ν). ⊠

Proof. Let $t_0 \geq 0$ and $t_0 \in (\tau_i, \tau_{i+1}]$ for some $i \geq 1$. We have the following notation for $k = 1, 2, \cdots$:

$$\tau_k \triangleq \begin{cases} \tau_{i+k}, & \text{if } \ t_0 \in (\tau_i, \tau_{i+1}), \\ \tau_{i+1+k}, & \text{if } \ t_0 = \tau_{i+1}. \end{cases} \tag{6.54}$$

1. (h_0, h)-practical stability
   Suppose that the comparison system (6.38) is practically stable with respect to $(\alpha(\mu), \beta(\nu))$, then $w_0 < \alpha(\mu)$ implies

$$w(t, t_0, w_0) < \beta(\nu), \qquad t \geq t_0, \tag{6.55}$$

where $w(t, t_0, w_0)$ is any a solution of (6.38) on $[t_0, \infty)$. Let us choose $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ such that $h_0(t_0, \boldsymbol{x}_0) < \mu$. Then from assumptions 2 and 5, we have

$$h(t_0, \boldsymbol{x}_0) \leq \phi(h_0(t_0, \boldsymbol{x}_0)) \leq \phi(\mu) < \nu. \tag{6.56}$$

We then have the following claim:
Claim 6.2.2: Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)$ be any a solution of (6.37) with $h_0(t_0, \boldsymbol{x}_0) < \mu$, we have

$$h(t, \boldsymbol{x}(t)) < \nu, \quad \forall t \geq t_0. \tag{6.57}$$

If Claim 6.2.2 is not true, then there are a $\tilde{\boldsymbol{u}}_1$ and a corresponding solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}_1)$ of (6.37) with $h_0(t_0, \boldsymbol{x}_0) < \mu$ and a $t^* > t_0$ such that $\tau_k < t^* \leq \tau_{k+1}$ for some k, satisfying

$$h(t^*, \boldsymbol{x}(t^*)) \geq \nu, \text{ and } h(t, \boldsymbol{x}(t)) < \nu, \quad t \in [t_0, \tau_k]. \tag{6.58}$$

From assumption 6 we know that there is a t_1 such that $\tau_k < t_1 \leq t^*$ and

$$\nu \leq h(t_1, \boldsymbol{x}(t_1)) < \rho. \tag{6.59}$$

Let $m(t) = V(t, \boldsymbol{x}(t))$, $t \in [t_0, t_1]$, and $w_0 = V(t_0, \boldsymbol{x}_0)$, then from assumption 3 we have

$$\begin{aligned} D^+ m(t) &\leq g(t, m(t), U(t, \tilde{\boldsymbol{u}}_1(t))), \qquad t \in [t_0, t_1], \quad t \neq \tau_i, \\ m(\tau_i^+) &\leq \psi_i(m(\tau_i)), \quad i = 1, 2, \cdots, k, \end{aligned} \tag{6.60}$$

from which and in view of $\tilde{\boldsymbol{u}}_1 \in \Omega$ and $g(\cdot, \cdot, v)$ is nondecreasing in v, we have

$$\begin{aligned} D^+ m(t) &\leq g(t, m(t), w_{\max}(t)), \qquad t \in [t_0, t_1], \quad t \neq \tau_i, \\ m(\tau_i^+) &\leq \psi_i(m(\tau_i)), \quad i = 1, 2, \cdots, k \end{aligned} \tag{6.61}$$

where $w_{\max}(t) = w_{\max}(t, t_0, w_0)$ is the maximal solution of (6.38). It then follows from Lemma 6.2.2 that

$$m(t) \leq w_{\max}(t), \quad t \in [t_0, t_1]. \tag{6.62}$$

Then from (6.55), (6.59), and (6.62) we have the following contradiction:

$$\beta(\nu) \leq V(t_1, \boldsymbol{x}(t_1)) \leq w_{\max}(t_1) < \beta(\nu). \tag{6.63}$$

Therefore, Claim 6.2.2 is true and we conclude that the impulsive control system (6.37) is (h_0, h)-practically stable.

2. (h_0, h)-strongly practical stability
Suppose that the comparison system (6.38) is strongly practically stable with respect to $(\alpha(\mu), \beta(\nu))$. This implies that system (6.37) is (h_0, h)-practically stable . We then know that $h_0(t_0, \boldsymbol{x}_0) < \mu$ implies

$$h(t, \boldsymbol{x}(t)) < \nu, \quad t \geq t_0, \tag{6.64}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{x}^*)$ is any solution of system (6.37) with $h_0(t_0, \boldsymbol{x}_0) < \mu$. Given $0 < b < \nu$ and a $T > 0$ and since system (6.38) is practically quasistable, if $\beta(b) > 0$ we then have

$$w_0 < \alpha(\mu) \ \text{ implies } \ w(t, t_0, w_0) < \beta(b), \quad t \geq t_0 + T. \tag{6.65}$$

Let us choose $(t_0, \boldsymbol{x}_0)$ such that $h_0(t_0, \boldsymbol{x}_0) < \mu$, then in view of (6.64), arguments leading to (6.63) yield

$$V(t, \boldsymbol{x}(t)) \leq \gamma(t, t_0, \alpha(h_0(t_0, \boldsymbol{x}_0))), \quad t \geq t_0, \tag{6.66}$$

from which and (6.65) we have

$$\beta(h(t, \boldsymbol{x}(t))) \leq V(t, \boldsymbol{x}(t)) < \beta(b), \quad t \geq t_0 + T, \tag{6.67}$$

from which we have

$$h(t, \boldsymbol{x}(t)) < b, \quad t \geq t_0 + T. \tag{6.68}$$

Thus, the system (6.37) is (h_0, h)-strongly practically stable.
3. Other cases can be proven similarly.

■

By choosing different types of functions we have the following corollary of Theorem 6.2.2.

Corollary 6.2.2.

1. *Assume that*

$$g(t, w, w) = 0, \quad \psi_k(w) = d_k w, \quad d_k \geq 0, \quad k = 1, 2, \cdots,$$

and if

$$\prod_{k=1}^{\infty} d_k < \infty$$

then the impulsive control system (6.37) is (h_0, h)*-uniformly practically stable.*

2. *Assume that*

$$\begin{aligned} g(t, w, w) &= \frac{\lambda(t)}{dt} w, \quad \lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+], \\ \psi_k(w) &= d_k w, \quad d_k \geq 0, \quad k = 1, 2, \cdots, \\ \lambda(\tau_k) + \ln d_k &\leq \lambda(\tau_{k-1}), \qquad k = 1, 2, \cdots, \end{aligned} \tag{6.69}$$

then the impulsive control system (6.37) is (h_0, h)*-practically stable.*

⊠

6.2.3 Controllability

Theorem 6.2.3. *Assume that*

1. $0 < \mu < \nu$ *are given and* $\rho > \nu$*. For a given* $v \in \mathcal{PC}$ *we define the following control set*

$$\Omega = \{\tilde{\boldsymbol{u}} \in \mathbb{R}^m | \ \ U(t, \tilde{\boldsymbol{u}}) \leq v(t), t \geq t_0\};$$

2. $h_0, h \in \mathcal{H}$ *and* $h(t, \boldsymbol{x}) \leq \phi(h_0(t, \boldsymbol{x}))$ *with* $\phi \in \mathcal{K}$ *whenever* $h_0(t, \boldsymbol{x}) < \mu$*;*
3. $V \in \mathcal{V}_0$ *and there exist* $\alpha, \beta \in \mathcal{K}$ *such that*

$$\begin{aligned} \beta(h(t, \boldsymbol{x})) &\leq V(t, \boldsymbol{x}), \qquad \text{if} \quad h(t, \boldsymbol{x}) < \rho, \\ V(t, \boldsymbol{x}) &\leq \alpha(h_0(t, \boldsymbol{x})), \qquad \text{if} \quad h_0(t, \boldsymbol{x}) < \mu; \end{aligned} \tag{6.70}$$

4. *for* $(t, \boldsymbol{x}) \in (\tau_{k-1}, \tau_k) \times \mathbb{R}^n$ *and* $\tilde{\boldsymbol{u}}(t) \in \Omega$*,*

$$D^+ V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}}(t))), \qquad \text{if} \quad h(t, \boldsymbol{x}) < \rho, \tag{6.71}$$

where $g(t, w, v)$ *is nondecreasing in* v *for each* $(t, w) \in \mathbb{R}_+ \times \mathbb{R}$ *and*

$$V(t, \boldsymbol{x}) \leq \alpha(h_0(t, \boldsymbol{x})), \qquad \text{if} \quad h_0(t, \boldsymbol{x}) < \mu; \tag{6.72}$$

5. $\phi(\mu) < \nu$*,* $\alpha(\mu) < \beta(\nu)$ *and* $h(t, \boldsymbol{x}) < \nu$ *implies* $h(t, \boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x})) < \rho$ *for all* $k = 1, 2, \cdots$*;*
6. *there exists a control function* $v \in \mathcal{PC}$ *such that any solution* $w(t) = w(t, t_0, w_0, w)$ *of the system (6.38) satisfies*

$$w_0 \leq \alpha(\mu) \ \ \text{implies} \ \ w(t) < \beta(\nu), \qquad t \geq t_0, \tag{6.73}$$

and

$$w(t_0 + T) \leq \beta(b), \ 0 < b < \nu, \ \text{for some } T = T(t_0, w_0) > 0. \tag{6.74}$$

Then there are admissible controls $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}(t) \in \Omega$ *such that*

1. *the system (6.37) is* (h_0, h)*-practically stable;*
2. *all solutions* $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})$ *starting in*

$$\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \ h(t, \boldsymbol{x}) < \mu, t \geq t_0\}$$

are transferred to the region

$$\Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \ h(t, \boldsymbol{x}) \leq b\}$$

in a finite time $T^* = T^*(t_0, \boldsymbol{x}_0) = T(t_0, V(t_0, \boldsymbol{x}_0))$*; namely, the system (6.37) is controllable.*

⊠

Proof. Let $h_0(t_0, \boldsymbol{x}_0) \leq \phi(h_0(t_0, \boldsymbol{x}_0)) < \mu$ and $U(t, \tilde{\boldsymbol{u}}(t)) \leq v(t), t \geq t_0$, then we have $h_0(t_0, \boldsymbol{x}_0) \leq \phi(h_0(t_0, \boldsymbol{x}_0)) < \nu$.

Then we have the following claim:

Claim 6.2.3: For any solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})$ of system (6.37), we have

$$h_0(t_0, \boldsymbol{x}_0) < \mu \ \text{ implies } \ h(t, \boldsymbol{x}(t)) < \nu, \qquad t \geq t_0. \tag{6.75}$$

If Claim 6.2.3 is false, then there exist a $\tilde{\boldsymbol{u}}_1 \in \Omega$ and a corresponding solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}_1)$ of system (6.37) and a $t^* > t_0$ such that $t^* \in (\tau_k, \tau_{k+1}]$ for some k, satisfying

$$h(t^*, \boldsymbol{x}(t^*)) \geq \nu \ \text{ and } \ h(t, \boldsymbol{x}(t)) < \nu, \qquad t \in [t_0, \tau_k]. \tag{6.76}$$

From assumption 5 we know that there exists a $t_1 \in (\tau_k, t^*]$ such that

$$h(t_1, \boldsymbol{x}(t_1)) \in [\nu, \rho). \tag{6.77}$$

Let $m(t) = V(t, \boldsymbol{x}(t))$ for $t \in [t_0, t_1]$ then from assumption 4 we have

$$\begin{aligned} D^+ m(t) &\leq g(t, m(t), v(t)), \qquad t \neq \tau_i, \quad t \in [t_0, t_1], \\ m(\tau_i^+) &\leq \psi_i(m(\tau_i)), \qquad i = 1, 2, \cdots, k, \end{aligned} \tag{6.78}$$

which implies by Theorem 6.2.1 the estimate

$$m(t) \leq \gamma(t, t_0, w_0, v), \qquad t \in [t_0, t_1], \tag{6.79}$$

with $m(t_0) \leq w_0$, where $\gamma(t, t_0, w_0, v)$ is the maximal solution of (6.38). Let us choose $w_0 = V(t_0, \boldsymbol{x}_0)$, then we can get from assumption 3 and (6.76)-(6.79) the relation

$$\beta(h(t, \boldsymbol{x}(t))) \leq V(t, \boldsymbol{x}(t)) \leq \gamma(t, t_0, w_0, v), \qquad t \in [t_0, t_1]. \tag{6.80}$$

Now we are led to the following contradiction, in view of (6.73) and (6.77),

$$\beta(\nu) \le \beta(h(t_1, \boldsymbol{x}(t_1))) \le \gamma(t, t_0, w_0, v) < \beta(\nu), \tag{6.81}$$

which proves that Claim 6.2.3 is true and the system (6.37) is (h_0, h)-practically stable.

Furthermore, (6.80) holds for all $t \ge t_0$, and from (6.74) we have

$$h(t_0 + T^*, \boldsymbol{x}(t_0 + T^*)) \le b, \tag{6.82}$$

where $T^* = T(t_0, V(t_0, \boldsymbol{x}_0))$. We finish the proof of controllability. ■

6.2.4 Examples

Let us consider the following comparison system:

$$\begin{aligned} \dot{w} &= a(t)w + b(t)v(t), \qquad t \ne \tau_k, \\ \varDelta w(\tau_k^+) &= (d_k - 1)w(\tau_k), \quad d_k \ge 0, \qquad k = 1, 2, \cdots, \\ w(t_0) &= w_0. \end{aligned} \tag{6.83}$$

where $a, b \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$. The solution of (6.83) is given by

$$w(t) = \begin{cases} \exp\left(\int_{t_0}^t a(s)ds\right) \int_{t_0}^t \exp\left(-\int_{t_0}^s a(\kappa)d\kappa\right) b(s)v(s)ds \\ +w_0 \exp\left(\int_{t_0}^t a(s)ds\right), \qquad t \in [t_0, \tau_1], \\ \exp\left(\int_{\tau_{k-1}}^t a(s)ds\right) \int_{\tau_{k-1}}^t \exp\left(-\int_{\tau_{k-1}}^s a(\kappa)d\kappa\right) b(s)v(s)ds \\ +d_{k-1}w(\tau_{k-1}) \exp\left(\int_{\tau_{k-1}}^t a(s)ds\right), \\ \quad t \in (\tau_{k-1}, \tau_k], \quad k = 2, 3, \cdots. \end{cases} \tag{6.84}$$

Given $0 < \beta(\mu) < \beta(\nu)$ and $0 < \beta(\mu_1) < \beta(\nu)$ and let us choose $v \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}]$ such that

$$\int_{\tau_{k-1}}^{\tau_k} \exp\left(-\int_{\tau_{k-1}}^s a(\kappa)d\kappa\right) b(s)v(s)ds \le \gamma_k.$$

If $a(t) \le 0$ for $t \ge t_0$ and

$$\beta(\mu) \prod_{k=1}^{\infty} d_k + \sum_{k=1}^{\infty} \gamma_k \prod_{i=k}^{\infty} d_i \le \beta(\nu),$$

therefore, if $w_0 < \beta(\mu)$ from (6.84) we have

$$w(t, t_0, w_0) < \beta(\nu), \qquad t \le t_0.$$

It means that system (6.83) is practically stable.

Furthermore, if there is a $T > 0$ such that $t_0 + T \ne \tau_k$ and

$$\exp\left(\int_{\tau_{i-1}}^{\tau_i} a(s)ds\right) \le \sigma_{i-1}, \quad i = 1, 2, \cdots, k+1,$$

and

$$\beta(\mu)\prod_{i=1}^{k} d_i\sigma_{i-1} + \sum_{i=1}^{k}\prod_{j=1}^{k} d_j\sigma_{j-1}\gamma_i + \sigma_{k+1}\gamma_{k+1} < \beta(\mu_1)$$

then

$$w(t_0+T, t_0, w_0) < \beta(\mu_1),$$

which shows that the system (6.83) is controllable.

6.3 Practical Stability of Linear Impulsive Control Systems

Consider the following linear nonautonomous impulsive system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \tilde{B}\tilde{\boldsymbol{u}} + \boldsymbol{\sigma}(t), \quad t \ne \tau_k, \\ \Delta\boldsymbol{x} &= B_k\boldsymbol{x}, \quad t = \tau_k, \quad k = 1, 2, \cdots, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \end{aligned} \qquad (6.85)}$$

where A and $\tilde{B}$ are $n \times n$ and $n \times m$ matrices, respectively. B_k is an $n \times n$ matrix for each k, and $\boldsymbol{\sigma} : \mathbb{R}_+ \to \mathbb{R}^n$ is piecewise continuous.

Theorem 6.3.1. *Assume that*

1. $0 < \mu < \nu$ *are given;*
2. $$\lim_{h\to 0}\frac{1}{h}(\|I + hA\| - 1) \le -a, \quad a > 0,$$
   $\|\tilde{B}\| = b$, $\|\boldsymbol{\sigma}(t)\| \le l$ *and* $\|B_k\| = b_k$ *for* $k = 1, 2, \cdots$;
3. $a - b = \delta > 0$ *and*
   $$\mu\prod_{k=1}^{\infty}(1 + b_k) + \frac{1}{\delta}\sum_{k=1}^{\infty}\prod_{i=k}^{\infty}(1 + b_i) + \frac{1}{\delta} < \nu.$$

Then the system (6.85) is practically stable. ⊠

Proof. Let us choose $V(t, \boldsymbol{x}) = \|\boldsymbol{x}\|$ and $h_0(t, \boldsymbol{x}) = h(t, \boldsymbol{x}) = \|\boldsymbol{x}\|$ then we have

$$g(t, w, w) = (-a + b)w + l \quad \text{and} \quad \psi_k(w) = (1 + b_k)w.$$

We then construct the following comparison system:

$$\begin{aligned} \dot{w} &= -\delta w + l, \quad t \neq \tau_k, \\ w(\tau_k^+) &= (1 + b_k) w(\tau_k), \quad k = 1, 2, \cdots, \\ w(t_0^+) &= w_0 \end{aligned} \tag{6.86}$$

whose solutions are given by

$$\begin{aligned} w(t, t_0, w_0) = w_0 \prod_{j=1}^{k} (1 + b_j) e^{-\delta(t - t_0)} + \frac{1}{\delta} \sum_{j=1}^{k} \prod_{i=j}^{k} (1 + b_i) e^{-\delta(t - t_j)} \\ - \frac{1}{\delta} \sum_{j=1}^{k} \prod_{i=j}^{k} (1 + b_i) e^{-\delta(t - t_{j-1})} + \frac{1}{\delta} (1 - e^{-\delta(t - t_k)}), \\ t \in (\tau_k, \tau_{k+1}]. \end{aligned} \tag{6.87}$$

From assumption 3 we know that $w_0 < \mu$ implies $w(t, t_0, w_0) < \nu$ for $t \geq t_0$. Thus, followed Theorem 6.2.2 the proof is complete. ■

6.4 Practical Stability in Terms of Multicomparison Systems

In this section we consider the practical stability of the following impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad t \neq \tau_k, \\ \Delta \boldsymbol{x} &= \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}}), \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k = 1, 2, \cdots \end{aligned} \qquad (6.88)}$$

where $\boldsymbol{f} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$ and $\boldsymbol{u}_k \in C[\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n], k = 1, 2, \cdots$.

Let us construct the following D comparison systems:

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}), \quad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k), \boldsymbol{w}(\tau_k)), \quad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \end{aligned} \tag{6.89}$$

where $\boldsymbol{w} \in \mathbb{R}^D$ and

1. $\boldsymbol{g} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+^D \times \mathbb{R}_+^D, \mathbb{R}^D]$, $\boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{v})$ is quasimonotone nondecreasing in $\boldsymbol{w}$ for each $(t, \boldsymbol{v})$ and nondecreasing in $\boldsymbol{v}$ for each $(t, \boldsymbol{w})$;
2. $\boldsymbol{\psi}_k : \mathbb{R}_+^D \times \mathbb{R}_+^D \to \mathbb{R}_+^D$ and $\boldsymbol{\psi}_k(\boldsymbol{w}, \boldsymbol{v})$ is nondecreasing in $(\boldsymbol{w}, \boldsymbol{v})$ for each k.

Let Ω be the admissible control input for $\tilde{\boldsymbol{u}}$ defined by

$$\Omega = \{\tilde{\boldsymbol{u}} \in \mathbb{R}^m | \ \ U(t, \tilde{\boldsymbol{u}}) \preceq \boldsymbol{w}_{\max}(t), t \geq t_0\},$$

where $U \in C[\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}_+^D]$ and $\boldsymbol{w}_{\max}(t)$ is the maximal solution of the comparison system (6.89).

The following Lemma can be found in [14].

Lemma 6.4.1. *Let us assume that*

1. $\boldsymbol{g} \in C[\mathbb{R}_+ \times \mathbb{R}_+^D \times \mathbb{R}_+^D, \mathbb{R}^D]$, $\boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{v})$ *is quasimonotone nondecreasing in* $\boldsymbol{w}$ *for each* $(t, \boldsymbol{v})$ *and nondecreasing in* $\boldsymbol{v}$ *for each* $(t, \boldsymbol{w})$*;*
2. $\boldsymbol{w}_{\max}(t) \geq 0$, $t \geq t_0$, *is the maximal solution of*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}), \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \geq 0, \quad t \in [t_0, \infty); \end{aligned} \tag{6.90}$$

3. $\boldsymbol{w}^*_{\max}(t)$ *is the maximal solution of*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}_{\max}), \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \geq 0, \quad t \in [t_0, \infty). \end{aligned} \tag{6.91}$$

Then $\boldsymbol{w}_{\max}(t) = \boldsymbol{w}^*_{\max}(t), t \geq t_0$. ⊠

The following lemma is Lemma 2.2 in [22]. The proof is straightforward from the classical results in Lemma 6.4.1.

Lemma 6.4.2. *Assume that*

1. $\boldsymbol{g} \in C[\mathbb{R}_+ \times \mathbb{R}_+^D \times \mathbb{R}_+^D, \mathbb{R}^D]$, $\boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{v})$ *is quasimonotone nondecreasing in* $\boldsymbol{w}$ *for each* $(t, \boldsymbol{v})$ *and nondecreasing in* $\boldsymbol{v}$ *for each* $(t, \boldsymbol{w})$*;*
2. $\boldsymbol{w}_{\max}(t) \geq 0$, $t \geq t_0$, *is the maximal solution of*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}), \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \geq 0, \quad t \in [t_0, \infty); \end{aligned} \tag{6.92}$$

3. $\boldsymbol{w}^*_{\max}(t)$ *is the maximal solution of*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}_{\max}), \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \geq 0, \quad t \in [t_0, \infty); \end{aligned} \tag{6.93}$$

4. $\boldsymbol{m}, \boldsymbol{v} \in C[\mathbb{R}_+, \mathbb{R}_+^D]$,

$$D^+\boldsymbol{m}(t) \preceq \boldsymbol{g}(t, \boldsymbol{m}(t), \boldsymbol{v}(t)), \quad t \geq t_0,$$

   $\boldsymbol{v}(t) \preceq \boldsymbol{w}_{\max}(t)$, $t \geq t_0$ *and* $\boldsymbol{m}(t_0) \preceq \boldsymbol{w}_0$.

Then $\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t) = \boldsymbol{w}^*_{\max}(t)$ *for* $t \geq t_0$. ⊠

Lemma 6.4.3. *Let us suppose that*

1. *$\boldsymbol{w}_{\max}(t)$ is the maximal solution of the comparison system (6.89) on $[t_0, \infty)$;*
2. *$\boldsymbol{w}^*_{\max}(t)$ is the maximal solution of the following system*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}_{\max}(t)), \qquad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k), \boldsymbol{w}_{\max}(\tau_k)), \qquad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \succeq 0, \end{aligned} \tag{6.94}$$

on $[t_0, \infty)$;

3. *$\boldsymbol{m}, \boldsymbol{v} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}_+^D]$ and*

$$\begin{aligned} D^+\boldsymbol{m}(t) &\preceq \boldsymbol{g}(t, \boldsymbol{m}(t), \boldsymbol{v}(t)), \qquad t \neq \tau_k, \\ \boldsymbol{m}(\tau_k^+) &\preceq \boldsymbol{\psi}_k(\boldsymbol{m}(\tau_k), \boldsymbol{v}(\tau_k)), \qquad k = 1, 2, \cdots; \end{aligned} \tag{6.95}$$

4. *$\boldsymbol{v}(t) \preceq \boldsymbol{w}_{\max}(t)$, $t \geq t_0$;*
5. *$\boldsymbol{m}(t_0) \preceq \boldsymbol{w}_0$.*

Then

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}^*_{\max}(t), \qquad t \geq t_0.$$

⊠

Proof. Let $\boldsymbol{w}_{\max 1}(t, t_0, \boldsymbol{w}_0)$ and $\boldsymbol{w}^*_{\max 1}(t, t_0, \boldsymbol{w}_0)$ be maximal solutions of (6.89) and (6.94) on $[t_0, \tau_1]$, respectively. It follows from Lemma 6.4.2 that

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}^*_{\max 1}(t, t_0, \boldsymbol{w}_0) = \boldsymbol{w}_{\max 1}(t, t_0, \boldsymbol{w}_0), \quad t \in [t_0, \tau_1]. \tag{6.96}$$

Then from (6.96), assumption 3 and assumptions on $\boldsymbol{\psi}_k$ we have

$$\begin{aligned} \boldsymbol{m}(\tau_1^+) &\preceq \boldsymbol{\psi}_1(\boldsymbol{m}(\tau_1), \boldsymbol{v}(\tau_1)) \\ &\preceq \boldsymbol{\psi}_1(\boldsymbol{w}_{\max 1}(\tau_1), \boldsymbol{w}_{\max 1}(\tau_1)) \\ &= \boldsymbol{\psi}_1(\boldsymbol{w}^*_{\max 1}(\tau_1), \boldsymbol{w}_{\max 1}(\tau_1)) \triangleq \boldsymbol{w}_1^+. \end{aligned} \tag{6.97}$$

Let $\boldsymbol{w}_{\max 2}(t, \tau_1, \boldsymbol{w}_1^+)$ and $\boldsymbol{w}^*_{\max 2}(t, \tau_1, \boldsymbol{w}_1^+)$ be maximal solutions of (6.89) and (6.94) on $[\tau_1, \tau_2]$, respectively. By using Lemma 6.4.2 again we have

$$\begin{aligned} \boldsymbol{m}(t) &\preceq \boldsymbol{w}^*_{\max 2}(t, \tau_1, \boldsymbol{w}_1^+) \\ &= \boldsymbol{w}_{\max 2}(t, \tau_1, \boldsymbol{w}_1^+), \quad t \in [\tau_1, \tau_2]. \end{aligned} \tag{6.98}$$

Let $\boldsymbol{w}_{\max k}(t, \tau_{k-1}, \boldsymbol{w}_{k-1}^+)$ and $\boldsymbol{w}^*_{\max k}(t, \tau_{k-1}, \boldsymbol{w}_{k-1}^+)$ be maximal solutions of (6.89) and (6.94) on $[\tau_{k-1}, \tau_k]$, respectively. And let

$$\boldsymbol{w}_{k-1}^+ = \boldsymbol{\psi}_{k-1}(\boldsymbol{w}_{\max(k-1)}(\tau_{k-1}, \tau_{k-2}, \boldsymbol{w}_{k-2}^+), \boldsymbol{w}_{\max(k-1)}(\tau_{k-1}, \tau_{k-2}, \boldsymbol{w}_{k-2}^+)),$$

then by repeating the same process we have

$$\begin{aligned} \boldsymbol{m}(t) &\preceq \boldsymbol{w}^*_{\max k}(t, \tau_{k-1}, \boldsymbol{w}_{k-1}^+) \\ &= \boldsymbol{w}_{\max k}(t, \tau_{k-1}, \boldsymbol{w}_{k-1}^+), \quad t \in [\tau_{k-1}, \tau_k]. \end{aligned} \tag{6.99}$$

Let us choose a solution of the comparison system (6.89) as

$$\boldsymbol{w}(t) = \begin{cases} \boldsymbol{w}_0, & t = t_0, \\ \boldsymbol{w}_{\max 1}(t, t_0, \boldsymbol{w}_0), & t \in (t_0, \tau_1], \\ \boldsymbol{w}_{\max 2}(t, \tau_1, \boldsymbol{w}_1^+), & t \in (\tau_1, \tau_2], \\ \vdots \\ \boldsymbol{w}_{\max k}(t, \tau_{k-1}, \boldsymbol{w}_{k-1}^+), & t \in (\tau_{k-1}, \tau_k], \\ \vdots \end{cases} \tag{6.100}$$

we have

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}(t), \quad t \geq t_0.$$

Because $\boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0)$ is the maximal solution of the comparison system (6.89), we have

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0), \quad t \geq t_0.$$

■

We have the following theorem:

Theorem 6.4.1. *Assume that*

1. $0 < \mu < \nu$ *are given;*
2. $\boldsymbol{V} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+^D]$, $\boldsymbol{V}(t, \boldsymbol{x}) = (V_1(t, \boldsymbol{x}), V_2(t, \boldsymbol{x}), \cdots, V_D(t, \boldsymbol{x}))^\top$ *is locally Lipschitzian in* $\boldsymbol{x}$, *and there exist* $\alpha, \beta \in \mathcal{K}[\mathbb{R}_+, \mathbb{R}_+]$ *such that*

$$\beta(\|\boldsymbol{x}\|) \leq \sum_{i=1}^{D} V_i(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|), \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho, \quad \rho > \nu;$$

3. *for* $(t, \boldsymbol{x}) \in (\tau_k, \tau_{k+1}] \times \mathcal{S}_\rho$ *and* $\tilde{\boldsymbol{u}} \in \Omega$ *we have*

$$D^+\boldsymbol{V}(t, \boldsymbol{x}) \preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}}));$$

4. $\boldsymbol{x} \in \mathcal{S}_\nu$ *implies* $\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}}) \in \mathcal{S}_\rho$ *and*

$$\boldsymbol{V}(\tau_k^+, \boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}(\tau_k, \boldsymbol{x}), U(\tau_k, \tilde{\boldsymbol{u}})), \quad \boldsymbol{x} \in \mathcal{S}_\rho;$$

5. $\alpha(\mu) < \beta(\nu)$;
6. $\boldsymbol{w}_{\max}(t)$ *is the maximal solution of the comparison system (6.89) on* $[t_0, \infty)$;
7. $\boldsymbol{w}^*_{\max}(t)$ *is the maximal solution of the following system*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{w}_{\max}(t)), \quad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k), \boldsymbol{w}_{\max}(\tau_k)), \quad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \succeq 0, \end{aligned} \tag{6.101}$$

on $[t_0, \infty)$.

Then the practical stability properties of the comparison system (6.89) with respect to $(\alpha(\mu), \beta(\nu))$ *imply those of system (6.88) with respect to* (μ, ν) *for every* $\tilde{\boldsymbol{u}} \in \Omega$. ⊠

Proof. 1. Practical stability

Let us suppose that system (6.89) is practically stable with respect to $(\alpha(\mu), \beta(\nu))$, then we have

$$\sum_{i=1}^{D} w_{i0} < \alpha(\mu) \text{ implies } \sum_{i=1}^{D} w_i(t, t_0, \boldsymbol{w}_0) < \beta(\nu), \;\; t \geq t_0, \tag{6.102}$$

where $\boldsymbol{w}(t, t_0, \boldsymbol{w}_0)$ is any solution of system (6.89) in $[t_0, \infty)$ and $\boldsymbol{w}_0 = \{w_{i0}\}_{i=0}^{D}$.

Suppose that $\|\boldsymbol{x}_0\| < \mu$ and we claim that $\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}^*)\| < \nu$ with $\boldsymbol{u}^* \in \Omega$. If this is not the case, then there exists a $\boldsymbol{u}_1 \in \Omega$ such that

$$\|\boldsymbol{x}(t_2, t_0, \boldsymbol{x}_0, \boldsymbol{u}_1)\| \geq \nu, \;\; \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}_1)\| < \nu, \tag{6.103}$$

where $t_2 \in (\tau_k, \tau_{k+1}]$ for some k and $t \in [t_0, \tau_k]$. From assumption 4 we know that there exists a $t_1 \in (\tau_k, t_2]$ such that

$$\nu \leq \|\boldsymbol{x}(t_1, t_0, \boldsymbol{x}_0, \boldsymbol{u}_1)\| < \rho. \tag{6.104}$$

We have $\|\boldsymbol{x}(t)\| < \rho, t \in [t_0, t_1]$. Let $\boldsymbol{m}(t) = \boldsymbol{V}(t, \boldsymbol{x}(t)), t \in [t_0, t_1]$, $\boldsymbol{w}(0) = \boldsymbol{V}(t_0, \boldsymbol{x}_0)$ and in view of assumption 3 we have

$$\begin{aligned} D^+\boldsymbol{m}(t) &\preceq \boldsymbol{g}(t, \boldsymbol{m}(t), U(t, \boldsymbol{u}_1(t))), \quad t \in [t_0, t_1], \qquad t \neq \tau_k, \\ \boldsymbol{m}(\tau_i^+) &\preceq \boldsymbol{\psi}_i(\boldsymbol{m}(\tau_i), U(\tau_i, \boldsymbol{u}_1(\tau_i))), \qquad i = 1, 2, \cdots, k, \\ \boldsymbol{m}(t_0) &= \boldsymbol{w}(0). \end{aligned} \tag{6.105}$$

Since $\boldsymbol{u}_1 \in \Omega$ and $\boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{v})$ is nondecreasing in $\boldsymbol{v}$, we have

$$\begin{aligned} D^+\boldsymbol{m}(t) &\preceq \boldsymbol{g}(t, \boldsymbol{m}(t), \boldsymbol{w}_{\max}(t)), \quad t \in [t_0, t_1], \qquad t \neq \tau_i, \\ \boldsymbol{m}(\tau_i^+) &\preceq \boldsymbol{\psi}_i(\boldsymbol{m}(\tau_i), \boldsymbol{w}_{\max}(\tau_i)), \qquad i = 1, 2, \cdots, k. \end{aligned} \tag{6.106}$$

Thus, by Lemma 6.4.3 we have

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t), \qquad t \in [t_0, t_1]. \tag{6.107}$$

Assumption 2 and (6.102), (6.104), (6.107) lead to the following contradiction:

$$\begin{aligned} &\beta(\nu) \leq \sum_{i=1}^{D} V_i(t_1, \boldsymbol{x}(t_1)) \leq \sum_{i=1}^{D} w_{\max i}(t_1) < \beta(\nu), \\ &\boldsymbol{w}_{\max} = \{w_{\max i}\}_{i=1}^{D} \end{aligned} \tag{6.108}$$

which proves that the impulsive control system (6.88) is practically stable.

2. Strongly practical stability
   Let us suppose that system (6.89) is strongly practically stable with respect to $(\alpha(\mu), \beta(\nu))$. Thus system (6.88) is practically stable and we have

$$\|\boldsymbol{x}_0\| < \mu \text{ implies } \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}^*)\| < \nu, \ t \geq t_0. \tag{6.109}$$

   Given $0 < \nu_1 < \nu$ and $T > 0$ and since system (6.89) is quasipractically stable, we have

$$\begin{aligned} &\sum_{i=1}^{D} w_{i0} < \alpha(\mu) \text{ implies} \\ &\sum_{i=1}^{D} w_i(t, t_0, \boldsymbol{w}_0) < \beta(\nu), \ t \geq t_0 + T. \end{aligned} \tag{6.110}$$

   Let us suppose that $\|\boldsymbol{x}_0\| \leq \mu$ and because of (6.109), arguments leading to (6.108) yield

$$\boldsymbol{V}(t, \boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0), \ t \geq t_0. \tag{6.111}$$

   Then in views of (6.110), (6.111) and assumption 2, it follows that

$$\begin{aligned} \beta(\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}^*)\|) &\leq \sum_{i=1}^{D} V_i(t, \boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}^*)) \\ &\leq \sum_{i=1}^{D} w_{\max i}(t, t_0, \boldsymbol{w}_0) < \beta(\nu_1), \\ \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \boldsymbol{u}^*)\| &< \nu_1, \quad t \geq t_0 + T. \end{aligned} \tag{6.112}$$

   Thus, system (6.88) is practically stable.
3. Other practical stability
   Proofs are similar.

■

Let $\boldsymbol{V}^{[1]}, \boldsymbol{V}^{[2]} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+^D]$, $\boldsymbol{V}^{[1]}(t, \boldsymbol{x})$ and $\boldsymbol{V}^{[2]}(t, \boldsymbol{x})$ are locally Lipschitzian in $\boldsymbol{x}$. And we have the following two sets of comparison systems:

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}_1(t, \boldsymbol{w}, \boldsymbol{w}), \quad t \neq \tau_k \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k), \boldsymbol{w}(\tau_k)), \quad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}(0) \geq 0, \end{aligned} \tag{6.113}$$

where $\boldsymbol{\psi}_k : \mathbb{R}_+^D \times \mathbb{R}_+^D \to \mathbb{R}_+^D$ and $\boldsymbol{g}_1 \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+^D \times \mathbb{R}_+^D, \mathbb{R}^D]$ and $\boldsymbol{g}_1(t, \boldsymbol{w}, \boldsymbol{v})$ is quasimonotone nondecreasing in $\boldsymbol{w}$ for each $(t, \boldsymbol{v})$ and monodecreasing in $\boldsymbol{v}$ for each $(t, \boldsymbol{w})$.

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{g}_2(t, \boldsymbol{z}, \boldsymbol{z}), \quad t \neq \tau_k \\ \boldsymbol{z}(\tau_k^+) &= \boldsymbol{\phi}_k(\boldsymbol{z}(\tau_k), \boldsymbol{z}(\tau_k)), \quad k = 1, 2, \cdots, \\ \boldsymbol{z}(t_0) &= \boldsymbol{z}(0) \geq 0, \end{aligned} \tag{6.114}$$

where $\boldsymbol{\phi}_k : \mathbb{R}_+^D \times \mathbb{R}_+^D \to \mathbb{R}_+^D$ and $\boldsymbol{g}_2 \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+^D \times \mathbb{R}_+^D, \mathbb{R}^D]$ and $\boldsymbol{g}_2(t, \boldsymbol{w}, \boldsymbol{v})$ is quasimonotone nondecreasing in $\boldsymbol{w}$ for each $(t, \boldsymbol{v})$ and monodecreasing in $\boldsymbol{v}$ for each $(t, \boldsymbol{w})$.

Theorem 6.4.2. *Assume that*

1. $0 < \mu < \nu$ *are given;*
2. *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\nu$,

$$\sum_{i=1}^{D} V_i^{[1]}(t, \boldsymbol{x}) \leq \alpha_1(t, \|\boldsymbol{x}\|)$$

and for $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \left(\overline{\mathcal{S}_\mu} \cap \mathcal{S}_\nu\right)$

$$\beta(\|\boldsymbol{x}\|) \leq \sum_{i=1}^{D} V_i^{[2]}(t, \boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|) + \sum_{i=1}^{D} V_i^{[1]}(t, \boldsymbol{x}),$$

where $\alpha_1 \in \mathcal{CK}[\mathbb{R}_+, \mathbb{R}_+]$ *and* $\alpha_2 \in \mathcal{K}[\mathbb{R}_+, \mathbb{R}_+]$*;*
3. *for* $(t, \boldsymbol{x}) \in (\tau_{k-1}, \tau_k] \times \mathcal{S}_\nu$ *and* $\tilde{\boldsymbol{u}} \in \Omega$

$$D^+ \boldsymbol{V}^{[1]}(t, \boldsymbol{x}) \preceq \boldsymbol{g}_1(t, \boldsymbol{V}^{[1]}(t, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}}));$$

4. $\|\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}})\| \leq \|\boldsymbol{x}\|$ *for* $\boldsymbol{x} \in \mathcal{S}_\nu$*;*
5. *for* $\boldsymbol{x} \in \mathcal{S}_\nu$,

$$\boldsymbol{V}_1(\tau_k^+, \boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}_1(\tau_k, \boldsymbol{x}), U(\tau_k, \tilde{\boldsymbol{u}})), \quad k = 1, 2, \cdots,$$

where $\boldsymbol{\psi}_k(\boldsymbol{w}, \boldsymbol{v})$ *is nondecreasing in* $(\boldsymbol{w}, \boldsymbol{v})$ *for all* $k = 1, 2, \cdots$*;*
6. *for* $(t, \boldsymbol{x}) \in (\tau_{k-1}, \tau_k] \times \left(\overline{\mathcal{S}_\mu} \cap \mathcal{S}_\nu\right)$ *and* $\tilde{\boldsymbol{u}} \in \Omega$,

$$D^+ \boldsymbol{V}^{[2]}(t, \boldsymbol{x}) \preceq \boldsymbol{g}_2(t, \boldsymbol{V}^{[2]}(t, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}}));$$

7. *for* $\boldsymbol{x} \in \overline{\mathcal{S}_\mu} \cap \mathcal{S}_\nu$

$$\boldsymbol{V}^{[2]}(\tau_k^+, \boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}})) \preceq \boldsymbol{\phi}_k(\boldsymbol{V}^{[2]}(\tau_k, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}})), \quad k = 1, 2, \cdots,$$

where $\boldsymbol{\phi}_k(\boldsymbol{w}, \boldsymbol{v})$ *is nondecreasing in* $(\boldsymbol{w}, \boldsymbol{v})$*;*
8. $\alpha_1(t_0, \mu) + \alpha_2(\mu) < \beta(\nu)$, *for all* $t_0 \in \mathbb{R}_+$*;*
9.

$$\sum_{i=1}^{D} w_i(0) < \alpha_1(t_0, \mu) \text{ implies } \sum_{i=1}^{D} w_i(t, t_0, \boldsymbol{w}_0) < \alpha_1(t_0, \mu)$$

for $t \geq t_0$ *with some* $t_0 \in \mathbb{R}_+$, *where* $\boldsymbol{w}(t, t_0, \boldsymbol{w}_0)$ *is any solution of (6.113) and*

$$\sum_{i=1}^{D} z_i(0) < \alpha_2(\mu) + \alpha_1(t_0, \mu) \text{ implies } \sum_{i=1}^{D} z_i(t, t_0, \boldsymbol{z}_0) < \beta(\nu)$$

for $t \geq t_0$ *with all* $t_0 \in \mathbb{R}_+$, *where* $\boldsymbol{z}(t, t_0, \boldsymbol{z}_0)$ *is any solution of (6.114).*

Then the impulsive control system (6.88) is practically stable. ⊠

Proof. If for any solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})$ of system (6.88)

$$\|\boldsymbol{x}_0\| < \mu \text{ implies } \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})\| < \nu, \quad t \geq t_0,$$

is false, then there exist a $\tilde{\boldsymbol{u}}^* = \tilde{\boldsymbol{u}}^*(t) \in \Omega$ and a corresponding solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)$, and owing to assumption 4, there should exist $t_2 > t_1 > t_0$ such that

$$\begin{aligned}
&\|\boldsymbol{x}(t_1, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| = \mu, \quad \|\boldsymbol{x}(t_2, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| = \nu, \\
&\mu \leq \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| \leq \nu, \\
&t_1 \in (\tau_i, \tau_{i+1}], \ t_2 \in (\tau_j, \tau_{j+1}], \ t \in [t_1, t_2].
\end{aligned} \tag{6.115}$$

Then in view of assumptions 3 and 5, we get by Lemma 6.4.3,

$$\boldsymbol{V}^{[1]}(t, \boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max 1}(t, t_0, \boldsymbol{V}^{[1]}(t_0, \boldsymbol{x}_0)), \quad t \in [t_0, t_2], \tag{6.116}$$

where $\boldsymbol{w}_{\max 1}(t, t_0, \boldsymbol{w}_0)$ is the maximal solution of (6.113) passing though $(t_0, \boldsymbol{w}_0)$. Based on assumptions 6 and 7 we have

$$\boldsymbol{V}^{[2]}(t, \boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max 2}(t, t_1, \boldsymbol{V}^{[2]}(t_1, \boldsymbol{x}(t_1))), \quad t \in [t_1, t_2], \tag{6.117}$$

where $\boldsymbol{w}_{\max 2}(t, t_1, \boldsymbol{z}_1)$ is the maximal solution of (6.114) passing through $(t_1, \boldsymbol{z}_1)$ and $\boldsymbol{z}_1 = \boldsymbol{V}^{[2]}(t_1, \boldsymbol{x}(t_1))$. Thus, from assumptions 2, 9, and (6.116) we get

$$\begin{aligned}
&\sum_{i=1}^{D} V_i^{[1]}(t_1, \boldsymbol{x}(t_1)) \leq \sum_{i=1}^{D} w_{\max 1i}(t_1, t_0, \boldsymbol{V}^{[1]}(t_0, \boldsymbol{x}_0)) < \alpha_1(t_0, \mu), \\
&\boldsymbol{w}_{\max 1} = \{w_{\max 1i}\}_{i=1}^{D}.
\end{aligned} \tag{6.118}$$

From (6.115), (6.118), and assumption 2, we get

$$\begin{aligned}
\sum_{i=1}^{D} V_i^{[2]}(t_1, \boldsymbol{x}(t_1)) &\leq \alpha_2(\|\boldsymbol{x}(t_1, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\|) + \sum_{i=1}^{D} V_i^{[1]}(t_1, \boldsymbol{x}(t_1)) \\
&< \alpha_2(\mu) + \alpha_1(t_0, \mu).
\end{aligned} \tag{6.119}$$

Thus, from (6.115), (6.117), (6.119), and assumptions 2 and 9, we get

$$\begin{aligned}\beta(\nu) = \beta(\|\boldsymbol{x}(t_2, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\|) &\leq \sum_{i=1}^{D} V_i^{[2]}(t_2, \boldsymbol{x}(t_2)) \\ &\leq \sum_{i=1}^{D} w_{\max 2i}(t_2, t_1, \boldsymbol{V}^{[2]}(t_1, \boldsymbol{x}(t_1))) < \beta(\nu), \\ &\boldsymbol{w}_{\max 2} = \{w_{\max 2i}\}_{i=1}^{D} \end{aligned} \tag{6.120}$$

which is a contradiction. ■

We then study the practical stability of the following impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq \tau_k, \quad t \in \mathbb{R}_+, \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k, \\ \boldsymbol{x}(t_0^+) &= \boldsymbol{x}_0, \quad k = 1, 2, \cdots, \end{aligned} \tag{6.121}}$$

where $\boldsymbol{x} \in \mathcal{S}_\rho$, $\rho > 0$. $\boldsymbol{f} \in C[\mathbb{R}_+ \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ satisfies $\boldsymbol{f}(t, 0) = 0$ for $t \in \mathbb{R}_+$ and there is a constant $L > 0$ such that $\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ for $t \in \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_{\rho_0}$, $U \in C[\mathbb{N} \times \mathcal{S}_{\rho_0}, \mathbb{R}^n]$ and $U(k, 0) = 0$ for all $k \in \mathbb{N}$.

For a $\varsigma > 0$ and $D < n$ let us denote

$$\Omega \triangleq \{\boldsymbol{w} \in \mathbb{R}^n \mid \|\boldsymbol{w}\| < \varsigma\}.$$

Let us construct the following D comparison systems:

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}), \quad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)), \quad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \end{aligned} \tag{6.122}$$

where $\boldsymbol{w} \in \Omega$, $t \in \mathbb{R}_+$ and

1. $\boldsymbol{g} \in C[(\tau_k, \tau_{k+1}] \times \Omega, \mathbb{R}^D]$, $\boldsymbol{g}(t, \boldsymbol{w})$ is quasimonotonically increasing in $\mathbb{R}_+ \times \Omega$, $\boldsymbol{g}(t, 0) = 0$ for $t \in \mathbb{R}_+$ and for any $\boldsymbol{q} \in \Omega$ the following limit exists:
$$\lim_{(t, \boldsymbol{w}) \to (\tau_k^+, \boldsymbol{q})} \boldsymbol{g}(t, \boldsymbol{w}) = \boldsymbol{g}(\tau_k^+, \boldsymbol{q});$$
2. $\boldsymbol{\psi}_k : \Omega \to \mathbb{R}^D$ is nondecreasing in Ω and $\boldsymbol{\psi}_k(0) = 0$ for each $k \in \mathbb{N}$. Let $\boldsymbol{e}_D \triangleq \mathrm{col}(1, 1, \cdots, 1) \in \mathbb{R}^D$, then there is such a $\varpi \in (0, \varsigma)$ that for $k \in \mathbb{N}$ we have
$$\boldsymbol{\psi}_k(\varpi \boldsymbol{e}_D) \prec \varsigma \boldsymbol{e}_D.$$

Theorem 6.4.3. *Assume that there are such an $\alpha \in \mathcal{K}_1$ and a $\beta \in \mathcal{K}_2$ that for $t \in \mathbb{R}_+$ and $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$*

$$\alpha(\|\boldsymbol{x}\|) \leq \max_{1\leq i\leq D} V_i(t,\boldsymbol{x}), \tag{6.123}$$

$$\boldsymbol{V}(t,\boldsymbol{x}) \preceq \beta(t,\|\boldsymbol{x}\|)\boldsymbol{e}_D, \tag{6.124}$$

$$\sup_{(t,\boldsymbol{x})\in\mathbb{R}_+\times\mathcal{S}_{\rho_0}} \|\boldsymbol{V}(t,\boldsymbol{x})\| < \varsigma, \tag{6.125}$$

$$D^+\boldsymbol{V}(t,\boldsymbol{x}) \preceq \boldsymbol{g}(t,V(t,\boldsymbol{x})), \quad t\neq\tau_k \tag{6.126}$$

$$\boldsymbol{V}(\tau_k^+,\boldsymbol{x}+U(k,\boldsymbol{x})) \preceq \boldsymbol{\psi}_k(V(\tau_k,\boldsymbol{x})), \quad k\in\mathbb{N}. \tag{6.127}$$

There are numbers μ_0,ν_0,μ and ν such that

$$\sup_{t_0\in\mathbb{R}_+} \beta(t_0,\mu) \leq \mu_0 < \nu_0 < \alpha(\nu), \quad 0<\mu<\nu, \quad \nu_0<\varpi. \tag{6.128}$$

Then we have the following conclusions:

1. *if the trivial solution of the comparison system (6.122) is practically stable with respect to (μ_0,ν_0), then the trivial solution of system (6.121) is practically stable with respect to (μ,ν);*
2. *Let λ_0 and λ be such that $0<\lambda_0<\nu_0$ and $\alpha^{-1}(\lambda_0)\leq\lambda<\nu$, then if the trivial solution of the comparison system (6.122) is contracting practically stable with respect to (λ_0,μ_0,ν_0), then the trivial solution of system (6.121) is contracting practically stable with respect to (λ,μ,ν).*

⊠

Proof.

♣ Let us prove the practical stability first. Let $\boldsymbol{x}(t)=\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ be a solution of system (6.121) such that $t_0\in\mathbb{R}_+$, $\boldsymbol{x}(t_0,t_0,\boldsymbol{x}_0)=\boldsymbol{x}_0$ and $\|\boldsymbol{x}_0\|<\mu$. Then we have

$$\boldsymbol{V}(t_0^+,\boldsymbol{x}_0) \preceq \beta(t_0,\|\boldsymbol{x}_0\|)\boldsymbol{e}_D \triangleq \boldsymbol{w}_0. \qquad \Leftarrow (6.124) \tag{6.129}$$

Let $\boldsymbol{w}_{\max}(t)=\boldsymbol{w}_{\max}(t,t_0,\boldsymbol{w}_0)$ be the maximal solution of system (6.122) with $\boldsymbol{w}_{\max}(t_0,t_0,\boldsymbol{w}_0)=\boldsymbol{w}_0$, then we have

$$\boldsymbol{V}(t,\boldsymbol{x}(t)) \preceq \boldsymbol{w}_{\max}(t). \tag{6.130}$$

It follows from (6.123) and (6.130) that

$$\begin{aligned}\alpha(\|\boldsymbol{x}(t)\|) &\leq \max_{1\leq i\leq D} V_i(t,\boldsymbol{x}(t)) \qquad \Leftarrow (6.123)\\ &\leq \max_{1\leq i\leq D} w_{\max i}(t) \qquad \Leftarrow (6.130)\end{aligned} \tag{6.131}$$

which leads to

$$(\|\boldsymbol{x}(t)\|) \leq \alpha^{-1}\left(\max_{1\leq i\leq D} w_{\max i}(t)\right). \tag{6.132}$$

Since the trivial solution of the comparison system (6.122) is practically stable with respect to (μ_0, ν_0) and

$$\begin{aligned} 0 \preceq \boldsymbol{w}_0 &\triangleq \beta(t_0, \|\boldsymbol{x}_0\|)\boldsymbol{e}_D \\ &< \beta(t_0, \mu)\boldsymbol{e}_D \qquad \Leftarrow (\|\boldsymbol{x}_0\| < \mu) \\ &\preceq \mu_0 \boldsymbol{e}_D, \qquad \Leftarrow (6.128) \end{aligned} \tag{6.133}$$

there is such a function $\gamma \in \mathcal{K}_2$ that

$$\boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0) \preceq \gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|))\boldsymbol{e}_D, \tag{6.134}$$

$$\gamma(t_0, \mu_0) < \nu_0 < \varpi. \tag{6.135}$$

Then from (6.132),(6.134) and (6.135) we have

$$\|\boldsymbol{x}(t)\| \leq \alpha^{-1}(\gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|))). \tag{6.136}$$

Let us define $\gamma_1(t_0, \|\boldsymbol{x}_0\|) \triangleq \alpha^{-1}(\gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|)))$, then we have

$$\begin{aligned} \gamma_1(t_0, \mu) &= \alpha^{-1}(\gamma(t_0, \beta(t_0, \mu))) \\ &< \alpha^{-1}(\gamma(t_0, \mu_0)) \qquad \Leftarrow (6.128) \\ &< \alpha^{-1}(\nu_0) \qquad \Leftarrow (6.135) \\ &< \alpha^{-1}(\alpha(\nu)) = \nu, \qquad \Leftarrow (6.128) \end{aligned} \tag{6.137}$$

and in view of $\gamma_1 \in \mathcal{K}_2$ we know that $\|\boldsymbol{x}(t)\| < \nu, t \geq t_0$ provided $\|\boldsymbol{x}_0\| < \mu$; namely, the trivial solution of system (6.121) is practically stable with respect to (μ, ν).

♣ Let us then prove the contracting practical stability . If the trivial solution of the comparison system (6.122) is contracting practically stable with respect to $(\lambda_0, \mu_0, \nu_0)$, then there are $\gamma \in \mathcal{K}_2$ and $\gamma_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for $t > t_0$ we have

$$\begin{aligned} &\boldsymbol{w}_{\max}(t, t_0, \beta(t_0, \|\boldsymbol{x}_0\|)) \preceq \gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|))\gamma_0(t)\boldsymbol{e}_D, \\ &\gamma(t_0, \mu_0)\gamma_0(t) < \nu_0, \\ &\gamma(t_0, \mu_0)\gamma_0(t_0 + s) < \lambda_0 \text{ for some } s > 0. \end{aligned} \tag{6.138}$$

It follows from (6.132) and (6.138) that

$$\|\boldsymbol{x}(t)\| \leq \alpha^{-1}(\gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|))\gamma_0(t)) = \gamma_1(t_0, \|\boldsymbol{x}_0\|)\gamma_2(t) \tag{6.139}$$

where

$$\gamma_1(t_0, \|\boldsymbol{x}_0\|) \triangleq \alpha^{-1}(\gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|)))$$

and

$$\gamma_2(t) = \begin{cases} \frac{\alpha^{-1}(\gamma(t_0, \beta(t_0, \|\boldsymbol{x}_0\|))\gamma_0(t))}{\gamma_1(t_0, \|\boldsymbol{x}_0\|)}, & \boldsymbol{x}_0 \neq 0, \\ 0, & \boldsymbol{x}_0 = 0. \end{cases}$$

Since $\gamma_1 \in \mathcal{K}_2$ and $\gamma_1(t_0, \mu) < \nu$ we have

$$\begin{aligned} \gamma_1(t_0,\mu)\gamma_2(t_0+s) &= \alpha^{-1}(\gamma_1(t_0,\beta(t_0,\mu))\gamma_0(t_0+s)) \\ &\leq \alpha^{-1}(\gamma_1(t_0,\mu_0)\gamma_0(t_0+s)) \qquad \Leftarrow (6.128) \\ &< \alpha^{-1}(\lambda_0) \leq \lambda. \qquad \Leftarrow (6.138) \end{aligned} \tag{6.140}$$

Therefore, the trivial solution of system (6.121) is contracting practically stable with respect to (λ, μ, ν). ■

Similarly we have the following theorem.

Theorem 6.4.4. *Assume that there are such an $\alpha \in \mathcal{K}_1$ and a $\beta \in \mathcal{K}_2$ that for $t \in \mathbb{R}_+$ and $\boldsymbol{x} \in \mathcal{S}_{\rho_0}$*

$$\begin{aligned} &\alpha(\|\boldsymbol{x}\|) \leq \max_{1\leq i\leq D} V_i(t,\boldsymbol{x}), \\ &\boldsymbol{V}(t,\boldsymbol{x}) \preceq \beta(\|\boldsymbol{x}\|)\boldsymbol{e}_D, \\ &\sup_{(t,\boldsymbol{x})\in\mathbb{R}_+\times\mathcal{S}_{\rho_0}} \|\boldsymbol{V}(t,\boldsymbol{x})\| < \varsigma, \\ &D^+\boldsymbol{V}(t,\boldsymbol{x}) \preceq \boldsymbol{g}(t,V(t,\boldsymbol{x})), \quad t \neq \tau_k \\ &\boldsymbol{V}(\tau_k^+, \boldsymbol{x}+U(k,\boldsymbol{x})) \preceq \boldsymbol{\psi}_k(V(\tau_k,\boldsymbol{x})), \quad k \in \mathbb{N}. \end{aligned} \tag{6.141}$$

There are numbers μ_0, ν_0, μ and ν such that

$$\beta(\mu) \leq \mu_0 < \nu_0 < \alpha(\nu), \quad 0 < \mu < \nu, \quad \nu_0 < \varpi.$$

Then we have the following conclusions:

1. *if the trivial solution of the comparison system (6.122) is uniformly practically stable with respect to (μ_0, ν_0), then the trivial solution of system (6.121) is uniformly practically stable with respect to (μ, ν);*
2. *Let λ_0 and λ be such that $0 < \lambda_0 < \nu_0$ and $\alpha^{-1}(\lambda_0) \leq \lambda < \nu$, then if the trivial solution of the comparison system (6.122) is contracting uniformly practically stable with respect to $(\lambda_0, \mu_0, \nu_0)$, then the trivial solution of system (6.121) is contracting uniformly practically stable with respect to (λ, μ, ν).*

⊠

Note 6.4.1. The above two theorems are adopted from [2] with revision. ♦

6.4.1 An Example

As a demonstrating example, let us consider the following system[22]:

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} \eta(t) & \zeta(t) \\ \zeta(t) & \eta(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} -\sin^2 t(x_1^3 + x_1 x_2^2) + ag(u_1, u_2) \\ -\sin^2 t(x_1^2 x_2 + x_2^3) + bg(u_1, u_2) \end{pmatrix}, \qquad t \neq \tau_k, \\
\Delta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{bmatrix} e_k & f_k \\ f_k & e_k \end{bmatrix} \begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \end{pmatrix} \\
&\quad + \begin{pmatrix} h_k g(u_1(\tau_k), u_2(\tau_k)) \\ i_k g(u_1(\tau_k), u_2(\tau_k)) \end{pmatrix}, \qquad t = \tau_k, \quad k = 1, 2, \cdots, \\
\begin{pmatrix} x_1(t_0^+) \\ x_2(t_0^+) \end{pmatrix} &= \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.
\end{aligned} \tag{6.142}
$$

where a, b, e_k, f_k, h_k, i_k are real constants, $g(u_1, u_2) \geq 0$, $\eta(t) \geq 0$, $\zeta(t) \geq 0$.

Let us choose

$$
V_1 = \frac{1}{2}(x_1 + x_2)^2, \qquad V_2 = \frac{1}{2}(x_1 - x_2)^2,
$$

then we have

$$
V_1 + V_2 = x_1^2 + x_2^2 = \|\boldsymbol{x}\|^2.
$$

Thus we can choose $\alpha(x) = (1+\epsilon)x^2$ and $\beta(x) = (1-\epsilon)x^2$ with a constant $\epsilon \in (0, 1]$. We then have

$$
\begin{aligned}
\dot{V}_1 &= (x_1 + x_2)^2(\eta(t) + \zeta(t)) - \sin^2 t(x_1 + x_2)^2(x_1^2 + x_2^2) \\
&\quad + (a+b)(x_1 + x_2)g(u_1, u_2) \\
&\leq (\eta(t) + \zeta(t))V_1 + |a+b| g(u_1, u_2)\sqrt{2V_1}.
\end{aligned} \tag{6.143}
$$

Similar we have

$$
\dot{V}_2 \leq (\eta(t) - \zeta(t))V_2 + |a-b| g(u_1, u_2)\sqrt{2V_2}. \tag{6.144}
$$

Then we have

$$
\begin{aligned}
V_1(\tau_k^+, \boldsymbol{x}_k(\tau_k^+)) &= \frac{1}{2}(x_1(\tau_k^+) + x_2(\tau_k^+))^2 \\
&= \frac{1}{2}[(e_k + f_k)(x_1 + x_2) + (h_k + i_k)g(u_1(\tau_k), u_2(\tau_k))]^2 \\
&\leq \frac{1}{2}[|e_k + f_k|\sqrt{2V_1} + |h_k + i_k| g(u_1(\tau_k), u_2(\tau_k))]^2
\end{aligned}
$$

and similarly

$$
V_2(\tau_k^+, \boldsymbol{x}_k(\tau_k^+)) \leq \frac{1}{2}[|e_k - f_k|\sqrt{2V_2} + |h_k - i_k| g(u_1(\tau_k), u_2(\tau_k))]^2.
$$

Let us define

$$
U_1(\tilde{\boldsymbol{u}}) \triangleq \frac{1}{2}[(a+b)g(u_1, u_2)]^2, \qquad U_2(\tilde{\boldsymbol{u}}) \triangleq \frac{1}{2}[(a-b)g(u_1, u_2)]^2 \tag{6.145}
$$

then we have

$$
\begin{aligned}
g_1(t,w,U) &= [\eta(t)+\zeta(t)]w_1+\sqrt{2w_1}\sqrt{2U_1(\tilde{\boldsymbol{u}})},\\
g_1(t,w,U) &= [\eta(t)-\zeta(t)]w_2+\sqrt{2w_1}\sqrt{2U_1(\tilde{\boldsymbol{u}})},\\
\boldsymbol{\psi}_k(w(\tau_k),U(\boldsymbol{u}(\tau_k))) &= \frac{1}{2}\left(|e_k+f_k|\sqrt{2w_1}+\left|\frac{h_k+i_k}{a+b}\right|\sqrt{2U_1(\boldsymbol{u}(\tau_k))}\right)^2,\\
\phi_k(w(\tau_k),U(\boldsymbol{u}(\tau_k))) &= \frac{1}{2}\left(|e_k-f_k|\sqrt{2w_2}+\left|\frac{h_k-i_k}{a-b}\right|\sqrt{2U_2(\boldsymbol{u}(\tau_k))}\right)^2,
\end{aligned}
$$

from which we have the following comparison system:

$$
\begin{aligned}
\dot{w}_1 &= g_1(t,w,w)=[\eta(t)+\zeta(t)+2]w_1 \triangleq \sigma(t)w_1, \quad t\neq\tau_k,\\
\dot{w}_2 &= g_2(t,w,w)=[\eta(t)-\zeta(t)+2]w_2 \triangleq \gamma(t)w_2, \quad t\neq\tau_k,\\
w_1(\tau_k^+) &= \boldsymbol{\psi}_k(w(\tau_k),w(\tau_k))\\
&= \left(|e_k+f_k|+\frac{h_k+i_k}{a+b}\right)^2 w_1(\tau_k^+) \triangleq c_k w_1(\tau_k^+), k=1,2,\cdots,\\
w_2(\tau_k^+) &= \phi_k(w(\tau_k),w(\tau_k))\\
&= \left(|e_k-f_k|+\frac{h_k-i_k}{a-b}\right)^2 w_2(\tau_k^+) \triangleq d_k w_2(\tau_k^+), k=1,2,\cdots,\\
w_1(t_0) &= \frac{1}{2}(x_1(0)+x_2(0))^2 \triangleq w_1(0),\\
w_2(t_0) &= \frac{1}{2}(x_1(0)-x_2(0))^2 \triangleq w_2(0).
\end{aligned}
\tag{6.146}
$$

It is easy to verify that g, $\boldsymbol{\psi}_k$, and ϕ_k satisfy the assumptions of Theorem 6.4.1. The solution to (6.146) is given as follows:

$$
\begin{aligned}
w_1(t) &= w_1(0)\prod_{0<\tau_k<t} c_k \exp\left\{\int_{t_0}^{t}\sigma(s)ds\right\},\\
w_2(t) &= w_2(0)\prod_{0<\tau_k<t} d_k \exp\left\{\int_{t_0}^{t}\gamma(s)ds\right\}, \qquad t\geq t_0
\end{aligned}
$$

and thus we have the admissible control set as

$$\Omega=\{\boldsymbol{u}|\ \ U(\boldsymbol{u})\leq w(t,t_0,w_0), t\geq t_0\}.$$

Given $0<\mu<\nu$, if

¥1 $0\leq c_k\leq \exp\left\{-\int_{\tau_k}^{\tau_{k+1}}\sigma(s)ds\right\}$ and $0\leq d_k\leq \exp\left\{-\int_{\tau_k}^{\tau_{k+1}}\gamma(s)ds\right\}$ for all $k=1,2,\cdots$;

¥2

$$\exp\left\{\int_{t_0}^{\tau_1}\sigma(s)ds\right\}<\frac{\beta(\nu)}{\alpha(\mu)} \quad\text{and}\quad \exp\left\{\int_{t_0}^{\tau_1}\gamma(s)ds\right\}<\frac{\beta(\nu)}{\alpha(\mu)}$$

then the comparison system (6.146) is practically stable. Furthermore, given $0 < \nu_1 < \nu$ and $T > 0$ where $t_0 + T \in (\tau_i, \tau_{i+1}]$ for some i, and ¥$_1$ and ¥$_2$ hold for all $k \neq i$, and

$$0 \leq c_i \leq \frac{\beta(\nu_1)}{\alpha(\mu)} \exp\left\{-\int_{t_0}^{\tau_1} \sigma(s)ds - \int_{\tau_i}^{\tau_{i+1}} \sigma(s)ds\right\}$$

and

$$0 \leq d_i \leq \frac{\beta(\nu_1)}{\alpha(\mu)} \exp\left\{-\int_{t_0}^{\tau_1} \gamma(s)ds - \int_{\tau_i}^{\tau_{i+1}} \gamma(s)ds\right\},$$

then the comparison system (6.146) is strongly practically stable.

6.5 Controllability in Terms of Multicomparison Systems

From Theorem 2.2 of [22] we have the following lemma.

Lemma 6.5.1. *Assume that*

1. $D^+\boldsymbol{m}(t) \preceq \boldsymbol{g}(t, \boldsymbol{m}(t)), t \neq \tau_k$, $\boldsymbol{m}(\tau_k^+) \preceq \boldsymbol{\psi}_k(\boldsymbol{m}(\tau_k))$, $k = 1, 2, \cdots$, *where* $\boldsymbol{g} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+^D, \mathbb{R}^D]$, $\boldsymbol{g}(t, \boldsymbol{w})$ *is quasimonotone nondecreasing in* $\boldsymbol{w}$, *and* $\boldsymbol{\psi}_k : \mathbb{R}_+^D \to \mathbb{R}_+^D$ *is nondecreasing for* $k = 1, 2, \cdots$;
2. $\boldsymbol{w}_{\max}(t)$ *is the maximal solution of*

$$\begin{aligned} \dot{\boldsymbol{w}} &= \boldsymbol{g}(t, \boldsymbol{w}), \qquad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)), \qquad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \geq 0, \end{aligned} \tag{6.147}$$

   on $[t_0, \infty)$.

Then $\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t)$, $t \geq t_0$, *provided* $\boldsymbol{m}(t_0) \preceq \boldsymbol{w}_0$. ⊠

Theorem 6.5.1. *Assume that*

1. $0 < \mu < \nu$ *and* $0 < \nu_1 < \nu$ *are given;*
2. $\boldsymbol{V} \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+^D, \mathbb{R}_+^D]$, $\boldsymbol{V}(t, \boldsymbol{x})$ *is locally Lipschitzian in* $\boldsymbol{x}$, $Q \in \mathcal{K}[\mathbb{R}_+^D, \mathbb{R}_+]$, *and for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho$, $\rho > \nu$,

$$\beta(\|\boldsymbol{x}\|) \leq Q(\boldsymbol{V}(t, \boldsymbol{x})) \leq \alpha(\|\boldsymbol{x}\|), \quad \alpha, \beta \in \mathcal{K};$$

3. *for* $(t, \boldsymbol{x}) \in (\tau_k, \tau_{k+1}] \times \mathcal{S}_\rho, k = 1, 2, \cdots$, *and* $\tilde{\boldsymbol{u}} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^m]$

$$D^+\boldsymbol{V}(t, \boldsymbol{x}) \preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x}), U(t, \tilde{\boldsymbol{u}}));$$

4. $\boldsymbol{x} \in \mathcal{S}_\nu$ *implies* $\boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}}) \in \mathcal{S}_\rho$ *and*

$$\boldsymbol{V}(\tau_k^+, \boldsymbol{x} + \boldsymbol{u}_k(\boldsymbol{x}, \tilde{\boldsymbol{u}})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}(\tau_k, \boldsymbol{x}), U(\tau_k, \tilde{\boldsymbol{u}})), \qquad k = 1, 2, \cdots,$$

   where $\boldsymbol{x} \in \mathcal{S}_\rho$ *and* $U \in C[\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}_+^D]$;

5. $\alpha(\mu) < \beta(\nu)$;
6. *given a* $T > 0$ *satisfying* $t_0 + T \neq \tau_k$, *there exists a* $\boldsymbol{v} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}_+^D]$ *such that any solution* $\boldsymbol{w}(t, t_0, \boldsymbol{w}_0, \boldsymbol{v}(t))$ *of*

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{g}(t, \boldsymbol{w}, \boldsymbol{v}(t)), \qquad t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) &= \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k), \boldsymbol{v}(\tau_k)), \qquad k = 1, 2, \cdots, \\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0, \end{aligned} \tag{6.148}$$

satisfies

$$Q(\boldsymbol{w}_0) < \alpha(\mu) \text{ implies } Q(\boldsymbol{w}(t, t_0, \boldsymbol{w}_0, \boldsymbol{v})) < \beta(\nu), \ t \geq t_0, \tag{6.149}$$

and

$$Q(\boldsymbol{w}(t_0 + T, t_0, \boldsymbol{w}_0, \boldsymbol{v})) \leq \beta(\nu_1). \tag{6.150}$$

Then there exists an admissible control $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}(t)$ *such that system (6.88) is practically stable and all solutions* $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})$ *starting in* $\Omega_1 = \{\boldsymbol{x} | \ \|\boldsymbol{x}\| < \mu\}$ *are transferred to the region* $\Omega_2 = \{\boldsymbol{x} | \ \|\boldsymbol{x}\| < \nu_1\}$ *in a finite time* T*; namely, system (6.88) is controllable.* ⊠

Proof. Assume that $\|\boldsymbol{x}_0\| < \mu$ and there is a $\boldsymbol{v} = \boldsymbol{v}(t)$ such that assumption 6 holds. Let $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}(t)$ be any control such that $U(t, \tilde{\boldsymbol{u}}) = \boldsymbol{v}(t), t \geq t_0$. If for any solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})$ of system (6.88)

$$\|\boldsymbol{x}_0\| < \mu \text{ implies } \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}})\| < \nu, \qquad t \geq t_0$$

is false, there should exist a $\tilde{\boldsymbol{u}}^*(t)$ such that $\boldsymbol{v}(t) = U(t, \tilde{\boldsymbol{u}}^*)$. Correspondingly, there should be a solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)$ and a $t^* > t_0, t^* \in (\tau_k, \tau_{k+1}]$ for some k such that

$$\|\boldsymbol{x}(t^*, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| \geq \nu, \ \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| < \nu, \ t \in [t_0, \tau_k]. \tag{6.151}$$

Then, from assumption 4 we know that there is a $t_1 \in (\tau_k, t^*]$, such that

$$\nu \leq \|\boldsymbol{x}(t_1, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\| < \rho. \tag{6.152}$$

Let $\boldsymbol{m}(t) = \boldsymbol{V}(t, \boldsymbol{x}(t))$ for $t \in [t_0, t_1]$ and $\boldsymbol{w}_0 = \boldsymbol{V}(t_0, \boldsymbol{x}_0)$, from assumption 3 and Lemma 6.5.1, we have

$$\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0, \boldsymbol{v}), \qquad t \in [t_0, t_1], \tag{6.153}$$

where $\boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0, \boldsymbol{v})$ is the maximal solution of (6.148). Then, from assumption 2 we have

$$\begin{aligned} Q(\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \tilde{\boldsymbol{u}}^*)\|) &\leq Q(\boldsymbol{V}(t, \boldsymbol{x})) \\ &\leq Q(\boldsymbol{w}_{\max}(t, t_0, \boldsymbol{w}_0, \boldsymbol{v})), \qquad t \in [t_0, t_1]. \end{aligned} \tag{6.154}$$

Thus, (6.149), (6.151), and (6.154) lead to the following contradiction:

$$\begin{aligned}\beta(\nu) &\le \beta(\|\boldsymbol{x}(t_1,t_0,\boldsymbol{x}_0,\tilde{\boldsymbol{u}}^*)\|) \le Q(\boldsymbol{V}(t_1,\boldsymbol{x}(t_1)))\\ &\le Q(\gamma(t_1,t_0,\boldsymbol{w}_0,\boldsymbol{v})) < \beta(\nu)\end{aligned} \tag{6.155}$$

which proves the practical stability of system (6.88).

Since (6.154) holds for all $t \ge t_0$ and therefore by using (6.150), we have

$$\beta(\|\boldsymbol{x}(t_0+T,t_0,\boldsymbol{x}_0,\tilde{\boldsymbol{u}}^*)\|) \le Q(\gamma(t_0+T,t_0,\boldsymbol{w}_0,\boldsymbol{v})) \le \beta(\nu_1) \tag{6.156}$$

and thus $\|\boldsymbol{x}(t_0+T,t_0,\boldsymbol{x}_0,\tilde{\boldsymbol{u}}^*)\| \le \nu_1$. This means that system (6.88) is controllable. ■

6.5.1 Examples

Let a time-varying linear type-II impulsive control system be

$$\begin{aligned}\dot{\boldsymbol{w}} &= A(t)\boldsymbol{w} + B(t)\tilde{\boldsymbol{u}}(t), \qquad t \ne \tau_k,\\ \Delta\boldsymbol{w}(\tau_k) &= \left(\sum_{i=1}^{m} d_i(\tau_k)\tilde{u}_i(\tau_k)\right)\boldsymbol{w}(\tau_k), \qquad k = 1,2,\cdots,\\ \boldsymbol{w}(t_0) &= \boldsymbol{w}_0 \ge 0,\end{aligned} \tag{6.157}$$

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ continuous matrices on $\mathbb{R}_+$, respectively. $\tilde{\boldsymbol{u}} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^m]$ and $d_i(\tau_k) \in \mathbb{R}$ for all k and i. Let $\Phi(t)$ be the fundamental matrix of (6.157), then use transformation $\boldsymbol{w} = \Phi(t)\boldsymbol{z}$ we have

$$\begin{aligned}\dot{\boldsymbol{z}} &= \Phi^{-1}(t)B(t)\tilde{\boldsymbol{u}}(t), \qquad t \ne \tau_k,\\ \Delta\boldsymbol{z}(\tau_k) &= \left(\sum_{i=1}^{m} d_i(\tau_k)\tilde{u}_i(\tau_k)\right)\boldsymbol{z}(\tau_k), \qquad k = 1,2,\cdots,\\ \boldsymbol{z}(t_0) &= \boldsymbol{w}_0.\end{aligned} \tag{6.158}$$

The solution of (6.158) is given by

$$\begin{aligned}\boldsymbol{z}(t) = &\prod_{t_0<\tau_k<t}\left(I + \sum_{i=1}^{m} d_i(\tau_k)\tilde{u}_i(\tau_k)\right)\boldsymbol{w}_0\\ &+ \int_{t_0}^{t}\prod_{s<\tau_k<t}\left(I + \sum_{i=1}^{m} d_i(\tau_k)\tilde{u}_i(\tau_k)\right)\Phi^{-1}(s)B(s)\tilde{\boldsymbol{u}}(s)ds,\\ &t \ge t_0.\end{aligned} \tag{6.159}$$

Let us choose $\tilde{\boldsymbol{u}}(\tau_k) = 0, k = 1,2,\cdots$, then from (6.159) we have

$$\boldsymbol{z}(t) = \boldsymbol{w}_0 + \int_{t_0}^{t} \Phi^{-1}(s)B(s)\tilde{\boldsymbol{u}}(s)ds. \tag{6.160}$$

Let us choose $\tilde{\boldsymbol{u}}(t) = B(t)^\top \Phi^{-1}(t)^\top \boldsymbol{h} + \boldsymbol{v}(t)$, $t \neq \tau_k$, with

$$\int_{t_0}^{\infty} B(s)^\top \Phi^{-1}(s)^\top \boldsymbol{v}(s) ds = 0$$

for some vector $\boldsymbol{h}$, we have

$$\boldsymbol{z}(t) = \boldsymbol{w}_0 + \int_{t_0}^{t} \Phi^{-1}(s) B(s)[B(s)^\top \Phi^{-1}(s)^\top \boldsymbol{h} + \boldsymbol{v}(s)] ds. \tag{6.161}$$

Given $0 < \mu < \nu$ and $0 < \nu_1 < \nu$ and choose $\boldsymbol{h}$ such that

$$\int_{t_0}^{\infty} \|\Phi^{-1}(s) B(s) B(s)^\top \Phi^{-1}(s)^\top\| ds \leq \frac{\nu - \mu}{\|\boldsymbol{h}\|}.$$

Then from (6.161) and if $\|\boldsymbol{w}_0\| < \mu$ we have

$$\|\boldsymbol{z}(t)\| < \mu + \frac{\nu - \mu}{\|\boldsymbol{h}\|} = \nu, \qquad t \geq t_0.$$

Since $\|\boldsymbol{w}(t)\| \leq \|\Phi(t)\| \|\boldsymbol{z}(t)\|$ for $t \geq t_0$ by definition, we know that if $\|\Phi(t)\| < 1$ for $t \geq t_0$, then system (6.157) is practically stable.

If for a given T satisfying $t_0 + T \neq \tau_k$ and $\|\Phi(t_0 + T)\| \leq \nu_1/\nu$, then $\boldsymbol{w}(t_0 + T) < \nu_1$. Therefore system (6.157) is controllable.

6.6 Impulsive Control of Nonautonomous Chaotic Systems

In this section we study impulsive control of nonautonomous chaotic systems by using practical stability.

6.6.1 Theory

Let a general nonautonomous chaotic system be

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}(t)) \tag{6.162}$$

where $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable. $\tilde{\boldsymbol{u}} : \mathbb{R}_+ \to \mathbb{R}^m$ is an external force which is independent of the system. Suppose that a discrete instant set $\{\tau_i\}$ satisfies

$$0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots, \ \tau_i \to \infty \text{ as } i \to \infty.$$

Let

$$\boldsymbol{u}_i(\boldsymbol{x}) = \Delta \boldsymbol{x}|_{t=\tau_i} \triangleq \boldsymbol{x}(\tau_i^+) - \boldsymbol{x}(\tau_i) \tag{6.163}$$

be change of state variable at instant τ_i, then an impulsively controlled system is given by

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \tilde{\boldsymbol{u}}(t)), \; t \neq \tau_i, \\ \Delta \boldsymbol{x} = \boldsymbol{u}_i(\boldsymbol{x}), \qquad t = \tau_i, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, \qquad t_0 \geq 0, \quad i = 1, 2, \cdots . \end{cases} \tag{6.164}$$

Definition 6.6.1. *Comparison system*
Let $V \in \mathcal{V}_0$ and assume that

$$\begin{cases} D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}), v(t)), \; t \neq \tau_i, \\ V(t, \boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})) \leq \psi_i(V(t, \boldsymbol{x})), \; t = \tau_i \end{cases} \tag{6.165}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing. Then the system

$$\begin{cases} \dot{w} = g(t, w, v(t)), \quad t \neq \tau_i, \\ w(\tau_i^+) = \psi_i(w(\tau_i)), \; i = 1, 2, \cdots , \\ w(t_0^+) = w_0 \geq 0 \end{cases} \tag{6.166}$$

is the comparison system of (6.164). ⊠

Let the set Ω be

$$\Omega = \{\tilde{\boldsymbol{u}} \in \mathbb{R}^m \,|\, \Gamma(t, \tilde{\boldsymbol{u}}) \leq \gamma(t), t \geq t_0\} \tag{6.167}$$

where $\Gamma \in C[\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}_+]$ and $\gamma(t)$ is the maximal solution of the comparison system (6.166).

Theorem 6.6.1. *Assume that*

1. *$g \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t, u, v)$ is nondecreasing in u for each (t, v) and nondecreasing in v for each (t, u);*
2. *ψ_i is nondecreasing for each i;*
3. *$0 < \mu < \nu$ (resp. $0 < \nu < \mu$) is given;*
4. *$V \in \mathcal{PC}[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$, and there exist $\alpha, \beta \in \mathcal{K}$ such that*

$$\begin{aligned} &\beta(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|), \\ &(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho, \qquad \rho > \nu; \end{aligned} \tag{6.168}$$

5. *for $(t, \boldsymbol{x}) \in (\tau_i, \tau_{i+1}] \times \mathcal{S}_\rho$ and $\tilde{\boldsymbol{u}}(t) \in \Omega$*

$$D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}), \Gamma(t, \tilde{\boldsymbol{u}})); \tag{6.169}$$

6. *$\boldsymbol{x} \in \mathcal{S}_\nu$ implies $\boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x}) \in \mathcal{S}_\rho$ and $V(\tau_i^+, \boldsymbol{x} + \boldsymbol{u}_i(\boldsymbol{x})) \leq \psi_i(V(t, \boldsymbol{x}))$, $\boldsymbol{x} \in \mathcal{S}_\rho$;*
7. *$\alpha(\mu) < \beta(\nu)$ (resp., $\alpha(\mu) > \beta(\nu)$).*

Then, the practical stability properties of the comparison system (6.166), with respect to $(\alpha(\mu), \beta(\nu))$*, imply the corresponding practical stability properties of system (6.164) with respect to* (μ, ν) *for every* $\tilde{\mathbf{u}}(t) \in \Omega$. ⊠

Proof. Set $h(t, \boldsymbol{x}) = h_0(t, \boldsymbol{x}) = \|x\|$, from Theorem 6.2.2 we immediately get the conclusion. ■

Remark 6.6.1. This theorem can also be viewed as a special case of Theorem 3.1 of [22]. Theorem 6.6.1 is very powerful because it translates the stable problem of an nth-order impulsive differential equation into that of a first-order impulsive differential equation. In many cases, this translation provides us an easy way to study the stability of a high-order impulsive differential equation. In next theorem, we present the stability criterion for a first-order impulsive differential equation, which is the general form of the comparison system of impulsively controlled chaotic systems we will study. ♦

Theorem 6.6.2. *Let* $\delta_{\max}$ *be*

$$\delta_{\max} \triangleq \sup_{i \in \mathbb{N}} \{\tau_{i+1} - \tau_i\}. \tag{6.170}$$

For a given (μ, ν)*,* $(0 < \mu < \nu)$ *or* $(0 < \nu < \mu)$*,*

1. *Let*

$$\begin{aligned} g(t, w, v) &= \phi w + \theta, \quad \phi > 0, \quad \theta > 0, \quad t \neq \tau_i, \\ \psi_i(w) &= d_i w, \quad d_i > 0, \quad t = \tau_i, i = 1, 2, \cdots, \\ w(t_0^+) &= w_0 \geq 0. \end{aligned} \tag{6.171}$$

2.

$$\begin{aligned} &\mu \prod_{i=1}^{\infty} d_i e^{\phi(t-t_0)} + \frac{\theta}{\phi} \sum_{j=1}^{\infty} \prod_{i=j}^{\infty} d_i \left| e^{\phi(t-\tau_j)} - e^{\phi(t-\tau_{j-1})} \right| \\ &+ \left| \frac{\theta}{\phi}(1 - e^{\phi \delta_{\max}}) \right| < \nu \end{aligned} \tag{6.172}$$

Then the system (6.171) is practically stable with respect to (μ, ν) *for every* $v(t)$. ⊠

Proof. The solution of the comparison system (6.171) is given by[27]

$$\begin{aligned}
w(t, t_0, w_0) = {} & w_0 \prod_{i=1}^{k} d_i e^{\phi(t-t_0)} + \frac{\theta}{\phi} \sum_{j=1}^{k} \prod_{i=j}^{k} d_i \left(e^{\phi(t-\tau_j)} - e^{\phi(t-\tau_{j-1})} \right) \\
& + \frac{\theta}{\phi}(1 - e^{\phi(t-\tau_k)}) \\
< {} & w_0 \prod_{i=1}^{k} d_i e^{\phi(t-t_0)} + \frac{\theta}{\phi} \sum_{j=1}^{k} \prod_{i=j}^{k} d_i \left| e^{\phi(t-\tau_j)} - e^{\phi(t-\tau_{j-1})} \right| \\
& + \left| \frac{\theta}{\phi}(1 - e^{\phi(t-\tau_k)}) \right|, \\
& t \in (\tau_k, \tau_{k+1}].
\end{aligned} \tag{6.173}$$

We have

$$\left| \frac{\theta}{\phi}(1 - e^{\phi(t-\tau_k)}) \right| < \left| \frac{\theta}{\phi}(1 - e^{\phi \Delta_{max}}) \right|. \tag{6.174}$$

Since $w_0 < \mu$ we have $w(t, t_0, w_0) < \nu$. ■

6.7 Examples

In this section we present some examples of impulsive control of nonautonomous chaotic systems by using practical stability.

6.7.1 Example 1

First, we study the following chaotic system:

$$\begin{cases} \dot{x} = y - f(x) \\ \dot{y} = -\epsilon x - \zeta y + \eta \sin(\omega t) \end{cases} \tag{6.175}$$

where $f(\cdot)$ is a piecewise linear function given by

$$f(x) = bx + \frac{1}{2}(a - b)(|x + 1| - |x - 1|) \tag{6.176}$$

where $a < b < 0$. We choose the parameters as: $\epsilon = 1$, $\zeta = 1.015$ and $\omega = 0.75$, $\eta = 0.15$, $a = -1.02$, and $b = -0.55$. The uncontrolled system is chaotic with a chaotic attractor shown in Fig. 6.1.

The impulsively controlled chaotic system is given by

$$\boxed{\begin{aligned}
&\left.\begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= -\epsilon x - \zeta y + \eta \sin(\omega t) \end{aligned}\right\}, \quad t \neq \tau_i, \\
&\left.\begin{aligned} x(\tau_i^+) &= d_i x(\tau_i) \\ y(\tau_i^+) &= d_i y(\tau_i) \end{aligned}\right\}, \quad d_i > 0, \quad t = \tau_i.
\end{aligned}} \tag{6.177}$$

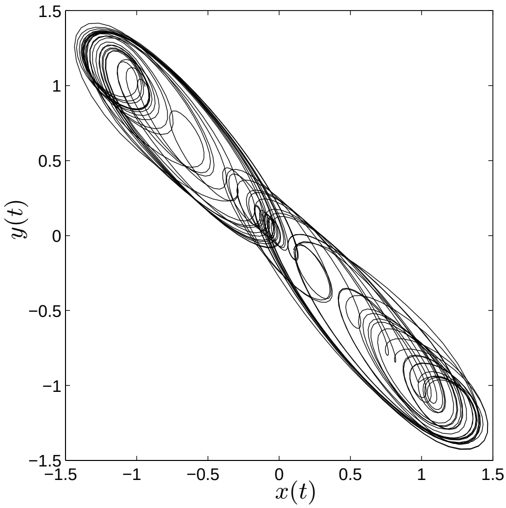


Fig. 6.1. The chaotic attractor of the uncontrolled chaotic system (6.175).

To give conditions for the practical stability of the impulsively controlled system (6.177), we first construct a comparison system for (6.177). We will use Proposition 6.7.1 to show that the practical stability of the comparison system implies that of (6.177). Finally, we will use Theorem 6.7.1 to show that the comparison system is practically stable.

Proposition 6.7.1. *The practical stability of impulsively controlled chaotic system (6.177) with respect to (μ, ν) is implied by that of the following comparison system*

$$\begin{aligned} g(t,w,v) &= \phi w + \theta, \phi = \max\{|\epsilon + |a||, |1-\zeta|\}, \quad \theta = |\eta|, \quad t \neq \tau_i, \\ \psi_i(w) &= d_i w, \quad d_i > 0, \quad t = \tau_i, i = 1, 2, \cdots, \\ w(t_0^+) &= w_0 \geq 0 \end{aligned} \tag{6.178}$$

with respect to $(\sqrt{2}\mu, \nu)$.

Proof. Choose a Lyapunov function as[53]

$$V(t, \boldsymbol{x}) = |x| + |y| \tag{6.179}$$

then we have

$$\begin{aligned} D^+V(t,\boldsymbol{x}) &= \dot{x}\mathrm{sgn}(x) + \dot{y}\mathrm{sgn}(y) \\ &= y\mathrm{sgn}(x) - f(x)\mathrm{sgn}(x) - \epsilon x\mathrm{sgn}(y) - \zeta|y| + \eta\mathrm{sgn}(y)\sin(\omega t) \\ &\leq |a||x| - \zeta|y| + (y\mathrm{sgn}(x) - \epsilon x\mathrm{sgn}(y)) + \eta\mathrm{sgn}(y)\sin(\omega t) \\ &\leq (\epsilon + |a|)|x| + (1-\zeta)|y| + |\eta| \\ &\leq \max\{|\epsilon + |a||, |1-\zeta|) + |\eta|. \end{aligned} \tag{6.180}$$

Notice that $-f(x)\text{sgn}(x) \le |a||x|$ due to $a < b < 0$, hence

$$g(t, w, v) = \phi w + \theta \tag{6.181}$$

where $\phi = \max\{|\epsilon + |a||, |1 - \zeta|)$ and $\theta = |\eta|$.

Since

$$V(\tau_i^+, \boldsymbol{x}(\tau_i^+)) = d_i V(\tau_i, \boldsymbol{x}(\tau_i)) \tag{6.182}$$

we have $\psi_i(w) = d_i w$.

It is well-known that $|x| + |y| \ge \sqrt{x^2 + y^2}$. Since

$$2(x^2 + y^2) = 2(|x|^2 + |y|^2) \ge |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \tag{6.183}$$

we have $|x| + |y| \le \sqrt{2}\sqrt{x^2 + y^2}$. We then have $\alpha(\chi) = \sqrt{2}\chi$ and $\beta(\chi) = \chi$. We find all conditions in Theorem 6.6.1 are satisfied. ■

Theorem 6.7.1. *Let $d_i = d > 0$ be constant for all $i = 1, 2, \cdots$, the impulses are equidistant with a fixed interval $\delta = \tau_{i+1} - \tau_i$ for all $i = 1, 2, \cdots$, and*

1.

$$\frac{1}{\delta}\ln(d) + \phi < 0, \tag{6.184}$$

2.

$$\frac{\theta}{\phi}|1 - e^{\phi\delta}| d e^{\phi\delta} \frac{1}{1 - de^{\phi\delta}} + \frac{\theta}{\phi}|1 - e^{\phi\delta}| < \nu,$$

i.e,

$$\frac{\theta}{\phi}\frac{|1 - e^{\phi\delta}|}{1 - de^{\phi\delta}} < \nu, \tag{6.185}$$

then the impulsively controlled chaotic system (6.177) is practically stable with respect to (μ, ν) for any $\mu < \infty$. ⊠

Proof. Let

$$\begin{aligned}
\xi(t) &= \left\lfloor \frac{t - t_0}{\delta} \right\rfloor \ln(d) + \phi(t - t_0) \\
&= \left\lfloor \frac{t - t_0}{\delta} \right\rfloor \ln(d) + \phi\left\{\delta \left\lfloor \frac{t - t_0}{\delta} \right\rfloor\right\} + \phi\left\{(t - t_0) - \delta\left\lfloor \frac{t - t_0}{\delta} \right\rfloor\right\} \\
&\le \left\lfloor \frac{t - t_0}{\delta} \right\rfloor (\ln(d) + \delta\phi) + \delta\phi
\end{aligned} \tag{6.186}$$

where $\lfloor \chi \rfloor$ denotes the largest integer less than or equal to χ.

Since $\frac{1}{\delta}\ln(d) + \phi < 0$ we have

$$\lim_{t\to\infty}\xi(t) = -\infty \tag{6.187}$$

then

$$\lim_{t\to\infty}\prod_{i=1}^{k} de^{\phi(t-t_0)} = \lim_{t\to\infty} e^{\lfloor \frac{t-t_0}{\delta}\rfloor \ln(d)+\phi(t-t_0)} = \lim_{t\to\infty} e^{\xi(t)} = 0, \tag{6.188}$$

from which we see that the first term in (6.172) becomes zero for any $\mu < \infty$.

For $t \in (\tau_k, \tau_{k+1}]$ we have

$$\begin{aligned}
&\lim_{k\to\infty}\frac{\theta}{\phi}\sum_{j=1}^{k}\prod_{i=j}^{k} d|e^{\phi(t-\tau_j)} - e^{\phi(t-\tau_{j-1})}| \\
&= \frac{\theta}{\phi}\lim_{k\to\infty}|1-e^{\phi\delta}|\sum_{j=1}^{k}\prod_{i=j}^{k} de^{\phi(t-\tau_j)} \\
&= \frac{\theta}{\phi}\lim_{k\to\infty}|1-e^{\phi\delta}|\sum_{j=1}^{k}\prod_{i=j}^{k} de^{\phi(\tau_{k+1}-\tau_j)} \\
&\le \frac{\theta}{\phi}|1-e^{\phi\delta}|\lim_{k\to\infty}\sum_{j=1}^{k} d^j e^{j\phi\delta} \\
&= \frac{\theta}{\phi}|1-e^{\phi\delta}|de^{\phi\delta}\frac{1}{1-de^{\phi\delta}}.
\end{aligned} \tag{6.189}$$

The last equation is satisfied if $de^{\phi\delta} < 1$, i.e., $\frac{1}{\delta}\ln(d) + \phi < 0$.

Since w_0 can be any finite value, it follows from Theorem 6.6.2 that the comparison system is practically stable with respect to $(\sqrt{2}\mu, \nu)$ for any $\mu < \infty$. It follows from Proposition 6.7.1 that the impulsively controlled chaotic system (6.177) is practically stable with respect to (μ, ν) for any $\mu < \infty$. ■

Remark 6.7.1. Since μ is used to decide the range of initial conditions, Theorem 6.7.1 guarantees that no matter how far the initial condition is away from the origin, the impulsively controlled chaotic system can be practically stabilized around the origin. ♦

6.7.2 Example 2

We then study the following chaotic system:

$$\begin{cases} \dot{x} = -x + \epsilon f(x) - \zeta f(y) + \eta \sin(\omega t) \\ \dot{y} = -y + \epsilon f(y) - \zeta f(x) \end{cases} \tag{6.190}$$

where $f(\cdot)$ is a piecewise linear function given by

$$f(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{6.191}$$

We choose the parameters as: $\epsilon = 2$, $\zeta = 1.2$, $\omega = \pi/2$, and $\eta = 4.04$. The uncontrolled system is chaotic with a chaotic attractor shown in Fig. 6.2.

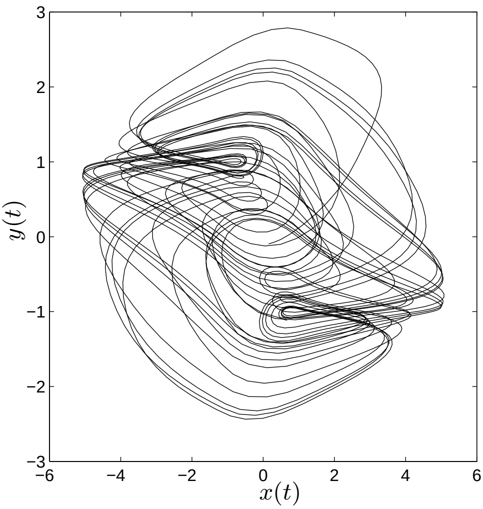


Fig. 6.2. The chaotic attractor of the uncontrolled chaotic system (6.190).

The impulsively controlled chaotic system is given by

$$\boxed{\begin{array}{ll} \left.\begin{array}{l} \dot{x} = -x + \epsilon f(x) - \zeta f(y) + \eta \sin(\omega t) \\ \dot{y} = -y + \epsilon f(y) - \zeta f(x) \end{array}\right\}, & t \neq \tau_i, \\ \left.\begin{array}{l} x(\tau_i^+) = d_i x(\tau_i) \\ y(\tau_i^+) = d_i y(\tau_i) \end{array}\right\}, \quad d_i > 0, \quad t = \tau_i. & \end{array}} \tag{6.192}$$

Theorem 6.7.2. *The practical stability of impulsively controlled chaotic system (6.192) with respect to (μ, ν) is implied by that of the following comparison system*

$$\begin{aligned} g(t, w, v) &= \phi w + \theta, \quad \phi = \epsilon - 1 + |\zeta|, \quad \theta = |\eta|, \quad t \neq \tau_i \\ \psi_i(w) &= d_i w, \quad d_i > 0, \quad t = \tau_i, \quad i = 1, 2, \cdots, \\ w(t_0^+) &= w_0 \geq 0 \end{aligned} \tag{6.193}$$

with respect to $(\sqrt{2}\mu, \nu)$. ⊠

Proof. Choose a Lyapunov function as[53]

$$V(t, \boldsymbol{x}) = |x| + |y|, \tag{6.194}$$

then we have

$$\begin{aligned} D^+V(t, \boldsymbol{x}) &= \dot{x}\mathrm{sgn}(x) + \dot{y}\mathrm{sgn}(y) \\ &= -|x| + \epsilon f(x)\mathrm{sgn}(x) - \zeta f(y)\mathrm{sgn}(x) + \eta\mathrm{sgn}(x)\sin(\omega t) \\ &\quad -|y| + \epsilon f(y)\mathrm{sgn}(y) + \zeta f(x)\mathrm{sgn}(y) \\ &\leq (\epsilon - 1 + |\zeta|)(|x| + |y|) + |\eta|, \end{aligned} \tag{6.195}$$

hence

$$g(t, w, v) = \phi w + \theta \tag{6.196}$$

where $\phi = \epsilon - 1 + |\zeta|$ and $\theta = |\eta|$. The rest part of the proof is the same as that in Proposition 6.7.1. ■

6.7.3 Example 3: Duffing's Oscillator

A Duffing's oscillator is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \epsilon y + \theta\cos(\omega t). \end{aligned} \tag{6.197}$$

We choose the parameters as: $\epsilon = 0.25$, $\theta = 0.3$ and $\omega = 1$. The uncontrolled system is a chaotic system whose chaotic attractor is shown in Fig. 6.3.

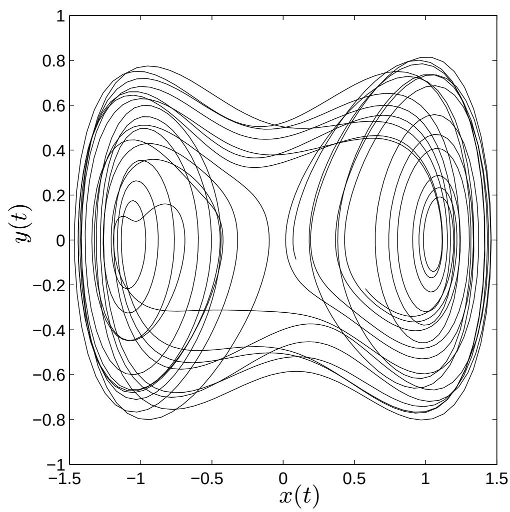


Fig. 6.3. The chaotic attractor of the uncontrolled Duffing's oscillator.

The impulsively controlled Duffing's oscillator is given by

$$
\left.\begin{array}{l}\dot{x} = y \\ \dot{y} = x - x^3 - \epsilon y + \theta\cos(\omega t)\end{array}\right\}, \quad t \neq \tau_i,
$$
$$
\left.\begin{array}{l}x(\tau_i^+) = d_i x(\tau_i) \\ y(\tau_i^+) = d_i y(\tau_i)\end{array}\right\}, \quad d_i > 0, \quad t = \tau_i. \tag{6.198}
$$

To give conditions for practical stability of the impulsively controlled system (6.198), we first construct a comparison system of (6.198). We then use Proposition 6.7.2 to show that the practical stability of the comparison system implies that of (6.198). Finally, we use Theorem 6.7.3 to show that the comparison system is practically stable.

Proposition 6.7.2. *The practical stability of impulsively controlled Duffing's oscillator in (6.198) with respect to (μ, ν) is implied by that of the following comparison system*

$$
\begin{aligned}
g(t, w, v) &= \phi w + \theta, \quad \phi = \max\{1 - \epsilon, 1 + x^2\}, \quad t \neq \tau_i, \\
\psi_i(w) &= d_i w, d_i > 0, \quad t = \tau_i, \quad i = 1, 2, \cdots, \\
w(t_0^+) &= w_0 \geq 0
\end{aligned} \tag{6.199}
$$

with respect to $(\sqrt{2}\mu, \nu)$.

Proof. Choose a Lyapunov function as[53]

$$
V(t, \boldsymbol{x}) = |x| + |y|. \tag{6.200}
$$

Then we have

$$
\begin{aligned}
D^+V(t, \boldsymbol{x}) &= \dot{x}\mathrm{sgn}(x) + \dot{y}\mathrm{sgn}(y) \\
&= y\mathrm{sgn}(x) + x\mathrm{sgn}(y) - x^3\mathrm{sgn}(y) - \epsilon|y| + \theta\sin(\omega t)\mathrm{sgn}(y) \\
&\leq (1 - \epsilon)|y| + (1 + x^2)|x| + |\theta\sin(\omega t)| \\
&\leq \max(1 - \epsilon, 1 + x^2)V(t, \boldsymbol{x}) + |\theta|.
\end{aligned} \tag{6.201}
$$

Hence

$$
g(t, w, v) = \max(1 - \epsilon, 1 + x^2)w + |\theta| = \phi w + \theta \tag{6.202}
$$

where $\phi = \max(1 - \epsilon, 1 + x^2)$ and $\theta = |\theta|$ because of $\theta > 0$. The rest part of the proof is the same as that in Proposition 6.7.1. ■

Theorem 6.7.3. *Let $d_i = d > 0$ be constant for all $i = 1, 2, \cdots$, the impulses are equidistant with a fixed interval $\delta = \tau_{i+1} - \tau_i$ for all $i = 1, 2, \cdots$, and*

1.

$$
\frac{1}{\delta}\ln(d) + \phi < 0, \tag{6.203}
$$

2.

$$\theta/\phi|1-e^{\phi\delta}|de^{\phi\delta}\frac{1}{1-de^{\phi\delta}}+\theta/\phi|1-e^{\phi\delta}|<\nu,$$

i.e,

$$\frac{\theta}{\phi}\frac{|1-e^{\phi\delta}|}{1-de^{\phi\delta}}<\nu, \tag{6.204}$$

then the impulsively controlled Duffing's oscillator is practically stable with respect to (μ,ν) *for any* $\mu<\infty$. ⊠

Proof. Since the comparison systems in (6.178) and (6.199) has the same form, the proof is similar to that of Theorem 6.7.1. ■

Note 6.7.1. The practical stability in terms of two measures is adopted from [19, 17]. The theory of practical stability in terms of multicomparison systems is adopted from [22]. The practical stability of control and synchronization of nonautonomous chaotic systems was presented in [50, 46, 30, 44]. ♦

7. Other Impulsive Control Strategies

In this chapter we study some other impulsive control strategies including: partial stability of impulsive control systems, and stability of integro-differential impulsive control systems.

7.1 Partial Stability of Impulsive Control

In this section we study the case when only a part of the state variables of an impulsive control system is stable; namely, partial stability. Let $\boldsymbol{x} = (x_1, \cdots, x_m, x_{m+1}, \cdot, x_n)^\top$ be an n-vector, then we can split $\boldsymbol{x}$ into an m-vector and an $(n-m)$-vector as $\boldsymbol{y} = (x_1, \cdots, x_m)^\top$ and $\boldsymbol{z} = (x_{m+1}, \cdots, x_n)^\top$. We have the following relation: $\boldsymbol{x} = (\boldsymbol{y}^\top, \boldsymbol{z}^\top)^\top$. Let us define

$$\begin{aligned} \mathcal{S}_\rho^m &\triangleq \{\boldsymbol{y} \in \mathbb{R}^m \mid \|\boldsymbol{y}\| < \rho\} \\ \maltese_\rho &\triangleq \mathcal{S}_\rho^m \times \mathbb{R}^{n-m}, \quad n > m. \end{aligned} \tag{7.1}$$

7.1.1 Control Impulses at Variable Time

In this section we study the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}), & t \neq \tau_k(\boldsymbol{x}), \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k(\boldsymbol{x}), \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & k = 1, 2, \cdots \end{cases} \tag{7.2}$$

where $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the uncontrolled plant, $\boldsymbol{u} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the additive continuous control input, $\tau_k : \mathbb{R}^n \to \mathbb{R}_+$, $U : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive control input. In this section, all results are based on the following assumptions:

1. $\boldsymbol{f} + \boldsymbol{u} \in C[\mathbb{R}_+ \times \maltese_\rho, \mathbb{R}^n]$ and satisfies Lipschitz condition in $\boldsymbol{x}$ with a constant L; namely,

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 199−217, 2001.

$$\|\boldsymbol{f}(t,\boldsymbol{x}_1)+\boldsymbol{u}(t,\boldsymbol{x}_1)-\boldsymbol{f}(t,\boldsymbol{x}_2)-\boldsymbol{u}(t,\boldsymbol{x}_2)\| \le L\|\boldsymbol{x}_1-\boldsymbol{x}_2\|,$$
$$t\in\mathbb{R}_+,\quad \boldsymbol{x}_1,\boldsymbol{x}_2\in\maltese_\rho \tag{7.3}$$

and $\boldsymbol{f}(t,0)+\boldsymbol{u}(t,0)=0$ for all $t\in\mathbb{R}_+$;

2. $U\in C[\mathbb{N}\times\maltese_\rho,\mathbb{R}^n]$, $U(k,0)=0$ for all $k\in\mathbb{N}$;
3. there is a $\rho_0\in(0,\rho)$ such that
$$\boldsymbol{x}\in\maltese_{\rho_0}\Rightarrow \boldsymbol{x}+U(k,\boldsymbol{x})\in\maltese_\rho,\quad k\in\mathbb{N};$$
4. $\tau_k: C[\maltese_\rho,\mathbb{R}_+]$, $k\in\mathbb{N}$ and for each $\boldsymbol{x}\in\maltese_\rho$ we have
$$0<\tau_1(\boldsymbol{x})<\tau_2(\boldsymbol{x})<\cdots<\tau_k(\boldsymbol{x})<\cdots,\quad \lim_{k\to\infty}\tau_k(\boldsymbol{x})=\infty;$$
5. every solution of system (7.2) hits any switching surface Σ_k at most once;
6. every solution $\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ of system (7.2), for which the estimate $\|\boldsymbol{y}(t,t_0,\boldsymbol{x}_0)\|\le \rho_0<\rho$ is valid for $t\in\mathfrak{J}^+(t_0,\boldsymbol{x}_0)$, is defined for all $t>t_0$.

Theorem 7.1.1. *Assume that*

1. $V\in\mathcal{V}_1$ *and* $\alpha\in\mathcal{K}$ *and*
$$\alpha(\|\boldsymbol{y}\|)\le V(t,\boldsymbol{x}) \text{ for } (t,\boldsymbol{y})\in\mathbb{R}_+\times\maltese_\rho, \tag{7.4}$$
$$\dot V(t,\boldsymbol{x})\le 0 \text{ for } (t,\boldsymbol{y})\in\mathfrak{G} \tag{7.5}$$
where the function $\dot V:\mathfrak{G}\to\mathbb{R}$ *is defined by*
$$\dot V(t,\boldsymbol{x})=\left\langle\frac{\partial V(t,\boldsymbol{x})}{\partial \boldsymbol{x}},\boldsymbol{f}(t,\boldsymbol{x})+\boldsymbol{u}(t,\boldsymbol{x})\right\rangle+\frac{\partial V(t,\boldsymbol{x})}{\partial t};$$
2.
$$V(t^+,\boldsymbol{x}+U(k,\boldsymbol{x}))\le V(t,\boldsymbol{x})$$
$$\text{for } (t,\boldsymbol{x})\in\Sigma_k\cap(\mathbb{R}_+\times\maltese_{\rho_0}),\quad k\in\mathbb{N}. \tag{7.6}$$

Then, the solution $\boldsymbol{x}\equiv 0$ *of system (7.2) is*

1. *partially stable with respect to* $\boldsymbol{y}$ *;*
2. *uniformly partially stable with respect to* $\boldsymbol{y}$ *if for some* $\beta\in\mathcal{K}$ *and for each* $(t,\boldsymbol{x})\in\mathbb{R}_+\times\maltese_\rho$ *we have*
$$V(t,\boldsymbol{x})\le\beta(\|\boldsymbol{x}\|). \tag{7.7}$$

$\boxtimes$

Proof. Let us first prove conclusion 1. Given $t_0\in\mathbb{R}_+$ and $\eta>0$, then there is a $\varepsilon=\varepsilon(t_0,\eta)>0$ such that
$$\sup_{\|\boldsymbol{x}\|<\varepsilon}|V(t_0^+,\boldsymbol{x})|<\min(\alpha(\eta),\alpha(\rho_0)).$$

Let $\boldsymbol{x}_0 \in \maltese_\rho$, $\|\boldsymbol{x}_0\| < \varepsilon$ and $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (7.2), then from (7.5) and (7.6) we know that $V(t, \boldsymbol{x})$ is nonincreasing in $\mathfrak{J}^+(t_0, \boldsymbol{x}_0)$. It follows from (7.4) that

$$\alpha(\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0)\|) \leq V(t, \boldsymbol{x}) \leq V(t_0, \boldsymbol{x}_0) < \min(\alpha(\rho_0), \alpha(\eta)) \tag{7.8}$$

from which and from the assumptions on α we have

$$\boldsymbol{y}(t, t_0, \boldsymbol{x}_0) < \min(\eta, \rho_0) \text{ for } t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0). \tag{7.9}$$

Since each solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of system (7.2), for which the estimate

$$\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0)\| \leq \rho_0 < \rho$$

is valid for $t \in \mathfrak{J}^+(t_0, \boldsymbol{x}_0)$, is defined in all $t > t_0$, it follows from (7.9) that $\mathfrak{J}^+(t_0, \boldsymbol{x}_0) = (t_0, \infty)$. Thus, (7.9) is true for $t > t_0$. And we conclude that the solution $\boldsymbol{x} \equiv 0$ of system (7.2) is partially stable with respect to $\boldsymbol{y}$.

We then prove conclusion 2. It follows from (7.7) that ε is independent of t_0 such that

$$\beta(\varepsilon) < \min(\alpha(\eta), \alpha(\rho_0)).$$

Therefore, the solution $\boldsymbol{x} \equiv 0$ of system (7.2) is uniformly partially stable with respect to $\boldsymbol{y}$. ■

Definition 7.1.1. *The set*

$$B(t_0) \triangleq \{\boldsymbol{x}_0 \mid \lim_{t\to\infty} \boldsymbol{y}(t, t_0, \boldsymbol{x}_0) = 0\}$$

is the basin of attraction of the origin with respect to $\boldsymbol{y}$ *at* t_0. ⊠

Definition 7.1.2. *Let* $t, s \in \mathbb{R}_+$, $V \in \mathcal{V}_0$, $\gamma \in \mathcal{K}$, *then we define the following set:*

$$\text{ß}_\gamma(t, s) \triangleq \{\boldsymbol{x} \in \maltese_\rho \mid V(t^+, \boldsymbol{x}) < \gamma(s)\}. \tag{7.10}$$

⊠

Theorem 7.1.2. *Assume that* $V \in \mathcal{V}_1$, $\alpha, \beta, \gamma \in \mathcal{K}$ *such that*

$$\alpha(\|\boldsymbol{y}\|) \leq V(t, \boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|) \text{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho, \tag{7.11}$$

$$\dot{V}(t, \boldsymbol{x}) \leq -\gamma(\|\boldsymbol{x}\|) \text{ for } (t, \boldsymbol{x}) \in \mathfrak{G}, \tag{7.12}$$

$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x}) \text{ for } (t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \maltese_{\rho_0}), k \in \mathbb{N}. \tag{7.13}$$

Then we have the following two conclusions:

1. $$\lim_{t\to\infty} \|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0\| = 0$$

   uniformly in $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \text{ß}_\gamma(t_0, \xi)$ *if* $\xi \in (0, \rho_0)$;

2. *the trivial solution of system (7.2) is uniformly asymptotically stable with respect to* $\boldsymbol{y}$*.*

$\boxtimes$

Proof. Let us prove conclusion 1 first. Given $\xi \in (0, \rho_0)$, then from (7.11) we have, for $t \in \mathbb{R}_+$

$$\ss_\alpha(t, \xi) = \{\boldsymbol{x} \in \maltese_\rho \mid V(t^+, \boldsymbol{x}) < \alpha(\xi)\} \subseteq \maltese_\xi \subset \maltese_\rho.$$

Given $t_0 \in \mathbb{R}_+$, $\boldsymbol{x}_0 \in \ss_\alpha(t_0, \xi)$ and let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of (7.2), then by using the similar procedure of the proof of Theorem 7.1.1 we have $\mathfrak{J}^+(t_0, \boldsymbol{x}_0) = (t_0, \infty)$ and $\boldsymbol{x}(t) \in \maltese_\xi$ for $t > t_0$. Given $\eta > 0$ then we can choose $\delta_1 = \delta_1(\eta) > 0$ and $\delta_2 = \delta_2(\eta) > 0$ such that $\beta(\delta_1) < \alpha(\eta)$ and $\delta_2 > \alpha(\xi)/\gamma(\delta_1)$. We then have the following claim:

Claim 7.1.2: There is a $t_1 \in (t_0, t_0 + \delta_2]$ such that $V(t_1, \boldsymbol{x}(t_1)) < \alpha(\eta)$.

If Claim 7.1.2 is not true, then for $t \in (t_0, t_0 + \delta_2]$ we have $V(t, \boldsymbol{x}(t)) \geq \alpha(\eta)$. It follows from (7.11) that $\|\boldsymbol{x}(t)\| \geq \delta_1$ for $t \in (t_0, t_0 + \delta_2]$. Then from (7.12) and (7.13) we have

$$\begin{aligned} V(t_0 + \delta_2, \boldsymbol{x}(t_0 + \delta_2)) &\leq V(t_0^+, \boldsymbol{x}_0) - \int_{t_0}^{t_0+\delta_2} \gamma(\|\boldsymbol{x}(s)\|) ds \\ &\leq \alpha(\xi) - \gamma(\delta_1)\delta_2 < 0 \end{aligned} \tag{7.14}$$

which is a contradiction to (7.11). Therefore, Claim 7.1.2 is true. From the proof of Theorem 7.1.1 we know that $V(t, \boldsymbol{x}(t))$ is nonincreasing, then for $t \geq t_0 + \delta_2$ we have

$$\alpha(\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0\|) \leq V(t, \boldsymbol{x}(t)) \leq V(t_1, \boldsymbol{x}(t_1)) < \alpha(\eta)$$

from which we know that for $t \geq t_0 + \delta_2$

$$\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0)\| \leq \eta.$$

This proves conclusion 1.

We are then going to prove conclusion 2. It follows from Theorem 7.1.1 that the trivial solution of system (7.2) is uniformly stable. Let us choose $\varepsilon > 0$ such that $\beta(\varepsilon) < \alpha(\xi)$ and in view of (7.11) we have $\{\boldsymbol{x} \in \maltese_\rho \mid \|\boldsymbol{x}\| < \varepsilon\} \subseteq \ss_\gamma(t, \xi)$. This means that the trivial solution of system (7.2) is uniformly attractive with respect to $\boldsymbol{y}$. ■

Theorem 7.1.3. *Assume that there are* $V, W \in \mathcal{V}_1$ *and* $\alpha, \beta, \gamma \in \mathcal{K}$ *such that*

1. *for* $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho$ *we have*

$$\alpha(\|\boldsymbol{y}\|) \leq V(t, \boldsymbol{x}), \quad \beta(\|\boldsymbol{y}\|) \leq W(t, \boldsymbol{x});$$

2. *for* $(t, \boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \maltese_{\rho_0})$, $k \in \mathbb{N}$, *we have*
$$V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq V(t, \boldsymbol{x})$$
and
$$W(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq W(t, \boldsymbol{x})$$
(resp. $W(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \geq W(t, \boldsymbol{x})$*);*
3. *for* $(t, \boldsymbol{x}) \in \mathfrak{G}$ *we have* $\dot{V}(t, \boldsymbol{x}) \leq -\gamma(W(t, \boldsymbol{x}))$*;*
4. $\dot{W}(t, \boldsymbol{x})$ *is bounded from above (resp. from below) on* $\mathfrak{G}$*.*

Then we have the following two conclusions:

1. *the basin of attraction with respect to* $\boldsymbol{y}$ *is* $B(t_0) \supseteq ß_\gamma(t_0, \xi)$ *if* $\xi \in (0, \rho_0)$ *and* $t_0 \in \mathbb{R}_+$*;*
2. *the trivial solution of system (7.2) is asymptotically stable with respect to* $\boldsymbol{y}$ *.*

⊠

Proof. Let us prove conclusion 1 first. Given $t_0 \in \mathbb{R}_+$, $\xi \in (0, \rho_0)$, $x_0 \in ß_\gamma(t_0, \xi)$ and let $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (7.2), then by using the same procedure in the proof of conclusion 1 of Theorem 7.1.2 we have
$$\mathfrak{J}^+(t_0, \boldsymbol{x}_0) = (t_0, \infty), \quad \boldsymbol{x}(t) \in \maltese_\xi \text{ for } t > t_0.$$
We then have the following claim:
Claim 7.1.3:
$$\lim_{t\to\infty} \|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0\| = 0 \text{ for } \boldsymbol{x}_0 \in ß_\gamma(t_0, \xi).$$

If Claim 7.1.3 is false, then there are $\boldsymbol{x}_0 \in ß_\gamma(t_0, \xi)$, $\varpi_1 > 0$ and $\varpi_2 > 0$ such that
$$\tau_i - \tau_{i-1} \geq \varpi_1 \text{ and } \|\boldsymbol{y}(\tau_i, t_0, \boldsymbol{x}_0)\| \geq \varpi_2 \text{ for } i \in \mathbb{N}.$$
Then it follows from assumption 1 that
$$|W(\tau_i^+, \boldsymbol{x}(\tau_i^+)| \geq \beta(\varpi_2), \quad i \in \mathbb{N} \tag{7.15}$$
where $\boldsymbol{x}(\tau_i^+) = \boldsymbol{x}(\tau_i) + U(i, \boldsymbol{x}(\tau_i))$. Let us choose the case that $\dot{W}(t, \boldsymbol{x})$ is bounded from above from assumption 4, then there is an $L > 0$ such that
$$\sup_{(t,\boldsymbol{x})\in\mathfrak{G}} \dot{W}(t, \boldsymbol{x}) < L. \tag{7.16}$$
Let us choose $\varpi_3 > 0$ such that
$$\varpi_3 < \min\left(\varpi_1, \frac{\beta(\varpi_2)}{2L}\right),$$
then it follows from (7.15), (7.16) and assumption 2 that

$$
\begin{aligned}
W(t,\boldsymbol{x}(t)) &\geq W(\tau_i^+,\boldsymbol{x}(\tau_i^+)) + \int_{\tau_i}^{t} \dot{W}(s,\boldsymbol{x}(s))ds \quad \Leftarrow \text{(assumption 2)} \\
&= W(\tau_i^+,\boldsymbol{x}(\tau_i^+)) - \int_{t}^{\tau_i} \dot{W}(s,\boldsymbol{x}(s))ds \\
&\geq \beta(\varpi_2) - L(\tau_i - t) \qquad \Leftarrow (7.15)\&(7.16) \\
&\geq \beta(\varpi_2) - L\varpi_3 \\
&> \beta(\varpi_2)/2, \quad t \in [\tau_i - \varpi_3, \tau_i].
\end{aligned}
\tag{7.17}
$$

Then, it follows from assumptions 2 and 3 and (7.17) that

$$
\begin{aligned}
0 &\leq V(\tau_k^+,\boldsymbol{x}(\tau_k^+)) \quad \Leftarrow \text{(assumption 2)} \\
&\leq V(t_0^+,\boldsymbol{x}_0) + \int_{t_0}^{\tau_k} \dot{V}(s,\boldsymbol{x}(s)))ds \\
&\leq V(t_0^+,\boldsymbol{x}_0) - \int_{t_0}^{\tau_k} \gamma(W(s,\boldsymbol{x}(s)))ds \quad \Leftarrow \text{(assumption 3)} \\
&\leq V(t_0^+,\boldsymbol{x}_0) - \sum_{i=1}^{k} \int_{\tau_i-\varpi_3}^{\tau_i} \gamma(W(s,\boldsymbol{x}(s)))ds \quad \Leftarrow (\gamma \in \mathcal{K}) \\
&\leq V(t_0^+,\boldsymbol{x}_0) - k\varpi_3\gamma(\beta(\varpi_2)/2) \quad \Leftarrow (7.17)
\end{aligned}
\tag{7.18}
$$

which leads to a contradiction if k is big enough. Therefore, Claim 7.1.3 is true. In the case when $\dot{W}(t,\boldsymbol{x})$ is bounded from below , by using similar process we can get the same conclusion. Thus, conclusion 1 has been proved.

We then prove conclusion 2. From Theorem 7.1.1 we know that the trivial solution of system (7.2) is stable with respect to $\boldsymbol{y}$. Since $\text{ß}_{t_0,\xi}$ is a neighborhood of $\boldsymbol{x} = 0$, it follows form conclusion 1 that the trivial solution of system (7.2) is attractive with respect to $\boldsymbol{y}$. Therefore, the trivial solution of system (7.2) is asymptotically stable with respect to $\boldsymbol{y}$. ■

Then from Theorem 7.1.3 we have the following corollary.

Corollary 7.1.1. *Assume that there are $V \in \mathcal{V}_1$ and $\alpha, \gamma \in \mathcal{K}$ such that*

$$
\begin{aligned}
&\alpha(\|\boldsymbol{y}\|) \leq V(t,\boldsymbol{x}) \text{ for } (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho, \\
&\dot{V}(t,\boldsymbol{x}) \leq -\gamma(V(t,\boldsymbol{x})) \text{ for } (t,\boldsymbol{x}) \in \mathfrak{G},
\end{aligned}
\tag{7.19}
$$

and

$$
\begin{aligned}
&V(t^+,\boldsymbol{x}+U(k,\boldsymbol{x})) \leq V(t,\boldsymbol{x}) \\
&\text{for } (t,\boldsymbol{x}) \in \Sigma_k \cap (\mathbb{R}_+ \times \maltese_{\rho_0}), \quad k \in \mathbb{N}.
\end{aligned}
\tag{7.20}
$$

Then we have the following two conclusions:

1. *the basin of attraction with respect to $\boldsymbol{y}$ is $B(t_0) \supseteq \text{ß}_\gamma(t_0,\xi)$ if $\xi \in (0,\rho_0)$ and $t_0 \in \mathbb{R}_+$;*
2. *the trivial solution of system (7.2) is asymptotically stable with respect to $\boldsymbol{y}$.*

⊠

7.1.2 Control Impulses at Fixed Time

We then use comparison system to study partial stability of the following impulsive control system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}), & t \neq \tau_k, \\ \Delta\boldsymbol{x} = U(k,\boldsymbol{x}), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & k = 1, 2, \cdots \end{cases} \tag{7.21}$$

where $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the uncontrolled plant, $\boldsymbol{u} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the additive continuous control input, and $U : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive control input. In this section, all results are based on the following assumptions.

1. $\boldsymbol{f} + \boldsymbol{u} \in C[\mathbb{R}_+ \times \maltese_\rho, \mathbb{R}^n]$ and satisfies Lipschitz condition in $\boldsymbol{x}$ with a constant L; namely,

$$\|\boldsymbol{f}(t,\boldsymbol{x}_1) + \boldsymbol{u}(t,\boldsymbol{x}_1) - \boldsymbol{f}(t,\boldsymbol{x}_2) - \boldsymbol{u}(t,\boldsymbol{x}_2)\| \le L\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$
$$t \in \mathbb{R}_+, \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in \maltese_\rho \tag{7.22}$$

   and $\boldsymbol{f}(t,0) + \boldsymbol{u}(t,0) = 0$ for all $t \in \mathbb{R}_+$;
2. $U \in C[\mathbb{N} \times \maltese_\rho, \mathbb{R}^n]$, $U(k,0) = 0$ for all $k \in \mathbb{N}$;
3. there is a $\rho_0 \in (0,\rho)$ such that

$$\boldsymbol{x} \in \maltese_{\rho_0} \Rightarrow \boldsymbol{x} + U(k,\boldsymbol{x}) \in \maltese_\rho, \quad k \in \mathbb{N};$$

4.

$$0 < \tau_1 < \tau_2 < \cdots \tau_k < \cdots, \quad \lim_{k\to\infty} \tau_k = \infty; \tag{7.23}$$

5. every solution $\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ of system (7.21), for which the estimate

$$\|\boldsymbol{y}(t,t_0,\boldsymbol{x}_0)\| \le \rho_0 < \rho \tag{7.24}$$

   is valid for $t \in \mathfrak{J}^+(t_0,\boldsymbol{x}_0)$, is defined for all $t > t_0$.

Let us set a comparison system of system (7.21) as

$$\begin{cases} \dot{\boldsymbol{w}} = \boldsymbol{g}(t,\boldsymbol{w}), & t \neq \tau_k, \\ \boldsymbol{w}(\tau_k^+) = \boldsymbol{\psi}_k(\boldsymbol{w}(\tau_k)), & t = \tau_k, \\ \boldsymbol{w}(t_0^+) = \boldsymbol{w}_0 \in \Omega, & k \in \mathbb{N} \end{cases} \tag{7.25}$$

where $\boldsymbol{g} : \mathbb{R}_+ \times \Omega \to \mathbb{R}^D$, $\boldsymbol{\psi}_k : \Omega \to \mathbb{R}^D$, and Ω is an open subset of $\mathbb{R}^D$. Let $\boldsymbol{w} = (w_1, \cdots, w_p, w_{p+1}, \cdots w_D)^\top$, $\boldsymbol{w}_y = (w_1, \cdots, w_p)^\top$ and $\boldsymbol{w}_z = (w_{p+1}, \cdots w_D)^\top$ and for every solution $\boldsymbol{w}(t,t_0,\boldsymbol{w}_0)$ of system (7.25), for which the estimate

$$\|\boldsymbol{w}_y(t,t_0,\boldsymbol{w}_0)\| \le \rho_0 < \rho \tag{7.26}$$

is valid for $t \in \mathfrak{J}^+(t_0,\boldsymbol{x}_0)$, is defined for all $t > t_0$.

Lemma 7.1.1. *Let us assume that*

1. $\boldsymbol{g} \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^D]$ *is quasimonotone increasing in* $\mathbb{R}_+ \times \Omega$*;*
2. $\boldsymbol{\psi}_k$, $k \in \mathbb{N}$ *are monotone increasing in* Ω*;*
3. $\boldsymbol{w}_{\max} : (t_0, T) \to \mathbb{R}^D$ *is the maximal solution of the comparison system (7.25) with* $\boldsymbol{w}_{\max}(t_0^+) = \boldsymbol{w}_0 \in \Omega$ *and* $t_0 \in \mathbb{R}_+$. $w(\tau_k^+) \in \Omega$ *if* $\tau_k \in (t_0, T)$*;*
4. $\boldsymbol{m} \in \mathcal{PC}[(t_0, T_1), \mathbb{R}^D]$, $T_1 \le T$ *such that*

$$
\begin{aligned}
&(t, \boldsymbol{m}(t)) \in \mathbb{R}_+ \times \Omega \text{ for } t \in (t_0, T_1), \\
&\boldsymbol{m}(\tau_k^+) \in \Omega \text{ provided } \tau_k \in (t_0, T_1), && (7.27) \\
&\boldsymbol{m}(t_0^+) \le \boldsymbol{w}_0, && (7.28) \\
&D^+\boldsymbol{m}(t) \le \boldsymbol{g}(t, \boldsymbol{m}(t)) \text{ for } t \in (t_0, T_1), \quad t \ne \tau_k, && (7.29) \\
&\boldsymbol{m}(\tau_k^+) \le \boldsymbol{\psi}_k(\boldsymbol{m}(\tau_k)) \text{ for } \tau_k \in (t_0, T_1). && (7.30)
\end{aligned}
$$

Then $\boldsymbol{m}(t) \le \boldsymbol{w}_{\max}(t)$ *for* $t \in (t_0, T_1)$. ⊠

Proof. Let us assume that $t_0 < \tau_1$ and let $t \in (t_0, \tau_1] \cap (t_0, T_1)$. Then it follows from classical comparison theorem and assumption 4 that $\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t)$ and $\boldsymbol{m}(\tau_1) \preceq \boldsymbol{w}_{\max}(\tau_1)$. From assumption 2 and (7.30) we have

$$
\begin{aligned}
\boldsymbol{m}(\tau_1^+) &\preceq \boldsymbol{\psi}_1(\boldsymbol{m}(\tau_1)) \\
&\preceq \boldsymbol{\psi}_1(\boldsymbol{w}_{\max}(\tau_1)) \triangleq \boldsymbol{w}_{\max}(\tau_1^+). \qquad (7.31)
\end{aligned}
$$

By using the similar procedure we have $\boldsymbol{m}(t) \preceq \boldsymbol{w}_{\max}(t)$ for $t \in (\tau_k, \tau_{k+1}] \cap (t_0, T_1)$, $k \in \mathbb{N}$. Therefore we finish the proof. ■

Theorem 7.1.4. *Let us assume that*

1. $\boldsymbol{V} : \mathbb{R}_+ \times \maltese_\rho \to \mathbb{R}^D$, $\boldsymbol{V} = \{V_i\}_{i=1}^D$, $\boldsymbol{V} \in \mathcal{V}_0$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and*

$$
\begin{aligned}
&\sup_{(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho} \|\boldsymbol{V}(t, \boldsymbol{x})\| < K \le \infty, \\
&\Omega = \{\boldsymbol{w} \in \mathbb{R}^D \mid \|\boldsymbol{w}\| < K\}; \qquad (7.32)
\end{aligned}
$$

2. $\boldsymbol{g}$ *is continuous, quasimonotone increasing in* $\mathbb{R}_+ \times \Omega$ *and* $\boldsymbol{g}(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*
3. $\boldsymbol{\psi}_k$, $k \in \mathbb{N}$ *are monotone increasing in* Ω *and* $\boldsymbol{\psi}_k(0) = 0$ *for all* $k \in \mathbb{N}$*;*
4. *for some* $\alpha \in \mathcal{K}$ *we have*

$$
\begin{aligned}
&\alpha(\|\boldsymbol{y}\|) \le \max_{1 \le i \le p} V_i(t, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho, && (7.33) \\
&D^+\boldsymbol{V}(t, \boldsymbol{x}) \preceq \boldsymbol{g}(t, \boldsymbol{V}(t, \boldsymbol{x})) \text{ for } t \ne \tau_k, \quad \boldsymbol{x} \in \maltese_\rho, && (7.34) \\
&\boldsymbol{V}(\tau_k^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}(\tau_k, \boldsymbol{x})) \\
&\text{for } \boldsymbol{x} \in \maltese_{\rho_0}, \quad k \in \mathbb{N}. && (7.35)
\end{aligned}
$$

Then we have the following conclusions:

1. *if the trivial solution of the comparison system (7.25) is stable with respect to $\boldsymbol{w}_y$ then the trivial solution of the impulsive control system (7.21) is stable with respect to $\boldsymbol{y}$;*
2. *if the trivial solution of the comparison system (7.25) is asymptotically stable with respect to $\boldsymbol{w}_y$ then the trivial solution of the impulsive control system (7.21) is asymptotically stable with respect to $\boldsymbol{y}$.*

$\boxtimes$

Proof. Let us prove conclusion 1 first. Given $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \maltese_\rho$ and let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be a solution of system (7.21). Let $\boldsymbol{m}(t) = \boldsymbol{V}(t, \boldsymbol{x}(t))$ be defined on (t_0, T_1) and the maximal solution $\boldsymbol{w}_{\max}(t) = \boldsymbol{w}_{\max}(t, t_0, \boldsymbol{V}(t_0^+, \boldsymbol{x}_0))$ of system (7.25) be defined in (t_0, T_2). Then from (7.33) and Lemma 7.1.1 we have for $t \in (t_0, T_1) \cap (t_0, T_2)$

$$\begin{aligned} \alpha(\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0)\|) &\leq \max_{1 \leq i \leq p} V_i(t, \boldsymbol{x}(t)) \\ &\leq \max_{1 \leq i \leq p} w_{\max i}(t, t_0, \boldsymbol{V}(t_0^+, \boldsymbol{x}_0)). \end{aligned} \tag{7.36}$$

Given $\eta > 0$ such that $\alpha(\eta) < K$ and $\eta < \rho$, since the trivial solution of system (7.25) is stable with respect to $\boldsymbol{w}_y$, there is a $\vartheta = \vartheta(t_0, \eta) > 0$ such that

$$\max_{1 \leq i \leq D} \|\boldsymbol{V}(t_0^+, \boldsymbol{x}_0)\| < \vartheta$$

implies

$$\max_{1 \leq i \leq p} \boldsymbol{w}_{\max i}(t, t_0, \boldsymbol{V}(t_0^+, \boldsymbol{x}_0)) < \alpha(\eta) \tag{7.37}$$

for $t \in (t_0, T_2)$. It follows from (7.26) that $T_2 = \infty$ and (7.37) is valid for all $t > t_0$.

From the properties of $\boldsymbol{V}$ we know that there is a $\varepsilon = \varepsilon(t_0, \eta) > 0$ such that $\varepsilon < \min(\alpha(\eta), \alpha(\rho_0))$ and if $\|\boldsymbol{x}_0\| < \varepsilon$ then

$$0 \leq \max_{1 \leq i \leq D} V_i(t_0^+, \boldsymbol{x}_0) < \vartheta. \tag{7.38}$$

It follows from (7.36), (7.37) and (7.38) that

$$\|\boldsymbol{y}(t, t_0, \boldsymbol{x}_0)\| < \eta \text{ provided } \|\boldsymbol{x}_0\| < \varepsilon, t \in (t_0, T_1). \tag{7.39}$$

From (7.24) we know that $T_1 = \infty$ and therefore (7.39) is valid for all $t > t_0$. We then finish the proof of conclusion 1.

The proof of conclusion 2 can be constructed in a similar way. ∎

Similarly, we have the following Theorem.

Theorem 7.1.5. *Let us assume that*

1. $\boldsymbol{V} : \mathbb{R}_+ \times \maltese_\rho \to \mathbb{R}^D$, $\boldsymbol{V} = \{V_i\}_{i=1}^D$, $\boldsymbol{V} \in \mathcal{V}_0$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and*

$$\begin{aligned} &\sup_{(t,\boldsymbol{x})\in\mathbb{R}_+\times\maltese_\rho} \|\boldsymbol{V}(t,\boldsymbol{x})\| < K \le \infty, \\ &\Omega = \{\boldsymbol{w} \in \mathbb{R}^D \mid \|\boldsymbol{w}\| < K\}; \end{aligned} \tag{7.40}$$

2. $\boldsymbol{g}$ *is continuous, quasimonotone increasing in* $\mathbb{R}_+ \times \Omega$ *and* $\boldsymbol{g}(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*
3. $\boldsymbol{\psi}_k$*,* $k \in \mathbb{N}$ *are monotone increasing in* Ω *and* $\boldsymbol{\psi}_k(0) = 0$ *for all* $k \in \mathbb{N}$*;*
4. *for some* $\alpha \in \mathcal{K}$ *we have*

$$\alpha(\|\boldsymbol{y}\|) \le \max_{1\le i\le p} V_i(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho \tag{7.41}$$

$$D^+\boldsymbol{V}(t,\boldsymbol{x}) \preceq \boldsymbol{g}(t,\boldsymbol{V}(t,\boldsymbol{x})) \text{ for } t \ne \tau_k, \quad \boldsymbol{x} \in \maltese_\rho, \tag{7.42}$$

$$\begin{aligned} &\boldsymbol{V}(\tau_k^+, \boldsymbol{x} + U(k,\boldsymbol{x})) \preceq \boldsymbol{\psi}_k(\boldsymbol{V}(\tau_k,\boldsymbol{x})) \\ &\text{for } \boldsymbol{x} \in \maltese_{\rho_0}, \quad k \in \mathbb{N}; \end{aligned} \tag{7.43}$$

5. *for some* $\beta \in \mathcal{K}$ *and for all* $(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \maltese_\rho$ *we have*

$$\max_{1\le i\le D} V_i(t,\boldsymbol{x}) \le \beta(\|\boldsymbol{x}\|). \tag{7.44}$$

Then we have the following conclusions:

1. *if the trivial solution of the comparison system (7.25) is uniformly stable with respect to* $\boldsymbol{w}_y$ *then the trivial solution of the impulsive control system (7.21) is uniformly stable with respect to* $\boldsymbol{y}$*;*
2. *if the trivial solution of the comparison system (7.25) is uniformly asymptotically stable with respect to* $\boldsymbol{w}_y$ *then the trivial solution of the impulsive control system (7.21) is uniformly asymptotically stable with respect to* $\boldsymbol{y}$*.*

⊠

7.2 Impulsive Control of Integro-differential Systems

In this section let us study a kind of impulsive control strategy that can be modeled by the following impulsive integro-differential equation:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x},T\boldsymbol{x}), & t \ne \tau_k, \\ \Delta\boldsymbol{x} = U(k,\boldsymbol{x}(\tau_k)), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & t_0 \ge 0, \quad k \in \mathbb{N} \end{cases}} \tag{7.45}$$

where $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $(\tau_k, \tau_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n$, $U : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$.

$$T\boldsymbol{x} = \int_{t_0}^{t} \boldsymbol{u}(t, s, \boldsymbol{x}(s))ds$$

where $\boldsymbol{u} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $(\tau_k, \tau_{k+1}] \times (\tau_k, \tau_{k+1}] \times \mathbb{R}^n$. We assume the existence and uniqueness of the solution of system (7.45) and

$$0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots, \quad \lim_{k\to\infty} \tau_k = \infty.$$

7.2.1 Comparison Results

Let us first present the following well-known comparison results for later use.

Lemma 7.2.1. *Let us assume that*

1. $g_0, g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$, $g_0(t, w) \le g(t, w)$, $w_{\max}(t, t_0, w_0)$ *is the right maximal solution of*

$$\dot{w} = g(t, w), \quad w(t_0) = w_0 \ge 0$$

on $[t_0, \infty)$ *and* $v_{\max}(t, t_1, v_0)$ *is the left maximal solution of*

$$\dot{v} = g_0(t, v), \quad v(t_1) = v_0 \ge 0$$

on $[t_0, t_1]$*;*

2. $V(t, \boldsymbol{x}) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and for* $t \ge t_0$, $\boldsymbol{x} \in \Xi$,

$$D_-V(t, \boldsymbol{x}) \le g(t, V(t, \boldsymbol{x}))$$

where

$$\Xi \triangleq \{\boldsymbol{x} \in C[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s)) \le v_{\max}(s, t, V(t, \boldsymbol{x})), s \in [t_0, t]\}$$

is the minimal class of functions along which $D_-V(t, \boldsymbol{x})$ *can be conveniently estimated.*

3. *let* $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ *be any solution of the following system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, T\boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad t_0 \ge 0 \tag{7.46}$$

on $[t_0, \infty)$ *such that* $V(t_0, \boldsymbol{x}_0) < w_0$.

Then, $V(t, \boldsymbol{x}(t)) < w_{\max}(t, t_0, w_0)$ *for* $t \ge t_0$. ⊠

Similarly, we can construct a comparison system as

$$\begin{cases} \dot{w} = g(t, w), & t \ne \tau_k, \\ w(\tau_k^+) = \psi_k(w(\tau_k)), & t = \tau_k, \\ w(t_0^+) = w_0 \ge 0, & t_0 \ge 0. \end{cases} \tag{7.47}$$

In this section we assume that $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $(\tau_k, \tau_{k+1}] \times \mathbb{R}_+$ and the limit

$$\lim_{(t,u)\to(\tau_k^+,w)} g(t,u) = g(\tau_k^+, w)$$

exists and let $w_{\max}(t, t_0, w_0)$ be the maximal solution of system (7.47) on $[t_0, \infty)$.

Theorem 7.2.1. *Let us assume that assumption 1 of Lemma 7.2.1 holds on each* $[\tau_k, \tau_{k+1}) \times \mathbb{R}_+$ *and*

1. $g_0 \in C[[\tau_k, \tau_{k+1}) \times \mathbb{R}_+, \mathbb{R}]$;
2. *let us define*

$$\Xi' \triangleq \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s)) \leq v_{\max}(s, t, V(t, \boldsymbol{x})), s \in [t_0, t]\}$$

then $V(t, \boldsymbol{x}) \in \mathcal{V}_0$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and for* $t \geq t_0$, $t \neq \tau_k$, *and* $\boldsymbol{x} \in \Xi'$ *we have*

$$D_-V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}));$$

3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x}))$ *for* $t = \tau_k$, $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ *is nonincreasing.*

Then, for any a solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, $t \in [t_0, \infty)$, *of system (7.45) we have*

$$V(t, \boldsymbol{x}(t)) \leq w_{\max}(t, t_0, w_0), t \geq t_0,$$

provided $V(t^+, \boldsymbol{x}_0) \leq w_0$. ⊠

Proof. Without loss of generality, let us suppose that $t_0 \in (\tau_k, \tau_{k+1}]$ for some $k \in \mathbb{N}$. Let $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$, $t \in [t_0, \infty)$ be any a solution of (7.45) and set $m(t) = V(t, \boldsymbol{x}(t))$. It follows from Lemma 7.2.1 that for $t \in (t_0, \tau_1]$ we have

$$m(t) \leq w_{\max 1}(t, t_0, w_0)$$

where $w_{\max 1}(t, t_0, w_0)$ is the maximal solution of the following differential equation

$$\dot{w} = g(t, w) \tag{7.48}$$

on $(t_0, \tau_1]$ such that $w_{\max 1}(t_0^+, t_0, w_0) = w_0$. Because $\psi_1(w)$ is nondecreasing in w and $m(\tau_1) \leq w_{\max 1}(\tau_1, t_0, w_0)$, then from assumption 3 we have

$$m(\tau_1^+) \leq \psi_1(w_{\max 1}(\tau_1, t_0, w_0)) \triangleq w_1^+.$$

Again, from Lemma 7.2.1 we have for $t \in (\tau_1, \tau_2]$

$$m(t) \leq w_{\max 2}(t, \tau_1, w_1^+)$$

where $w_{\max 2}(t, \tau_1, w_1^+)$ is the maximal solution of (7.48) on $(\tau_1, \tau_2]$ such that

$$w_{\max 2}(\tau_1^+, \tau_1, w_1^+) = w_1^+.$$

By repeating the same process we have

$$m(t) \le w_{\max(k+1)}(t, \tau_k, w_k^+), \quad t \in (\tau_k, \tau_{k+1}]$$

where $w_{\max(k+1)}(t, \tau_k, w_k^+)$ is the maximal solution of (7.48) on $(\tau_k, \tau_{k+1}]$ such that

$$w_{\max(k+1)}(\tau_k^+, \tau_k, w_k^+) = w_k^+.$$

Therefore if we choose the following solution of (7.47)

$$w(t) = \begin{cases} w_0, & t = t_0, \\ w_{\max 1}(t, t_0, w_0), & t \in (t_0, \tau_1], \\ w_{\max 2}(t, \tau_1, w_1^+), & t \in (\tau_1, \tau_2], \\ \vdots & \\ w_{\max(k+1)}(t, \tau_k, w_k^+), & t \in (\tau_k, \tau_{k+1}], \\ \vdots & \end{cases} \tag{7.49}$$

then we have

$$m(t) \le w(t), \quad t \ge t_0.$$

Because $w_{\max}(t, t_0, w_0)$ is the maximal solution of (7.47), we have

$$m(t) \le w_{\max}(t, t_0, w_0), \quad t \ge t_0.$$

■

Corollary 7.2.1. *In Theorem 7.2.1, let us assume that*

1. $g_0(t, w) = g(t, w) = 0$ *and* $\psi_k(w) = w$ *for all* $k \in \mathbb{N}$*, then* $V(t, \boldsymbol{x}(t))$ *is nonincreasing in* t *and* $V(t, \boldsymbol{x}(t)) \le V(t_0^+, \boldsymbol{x}_0)$ *for* $t \ge t_0$*;*
2. $g_0(t, w) = g(t, w) = 0$ *and* $\psi_k(w) = d_k w$*,* $d_k \ge 0$ *for all* $k \in \mathbb{N}$*, then*
$$V(t, \boldsymbol{x}(t)) \le V(t_0^+, \boldsymbol{x}_0) \prod_{t_0 < \tau_k < t} d_k, \quad t \ge t_0;$$
3. $g_0(t, w) = 0$*,* $g(t, w) = \dot{\lambda}(t) w$*,* $\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$*,* $\dot{\lambda}(t) \ge 0$ *and* $\psi_k(w) = d_k w$*,* $d_k \ge 0$ *for all* $k \in \mathbb{N}$*, then*
$$V(t, \boldsymbol{x}(t)) \le e^{\lambda(t) - \lambda(t_0)} V(t_0^+, \boldsymbol{x}_0) \prod_{t_0 < \tau_k < t} d_k, \quad t \ge t_0;$$
4. $$g_0(t, w) = g(t, w) = -\frac{\dot{\lambda}(t)}{\lambda(t)} w,$$
where $\lambda(t) > 0$ *is continuously differentiable on* $\mathbb{R}_+$*,*

$$\lim_{t\to\infty} A(t) = \infty$$

and $\psi_k(w) = d_k w,\ d_k \geq 0$ *for all* $k \in \mathbb{N},$ *then*

$$V(t, \boldsymbol{x}(t)) \leq \frac{\lambda(t_0)}{\lambda(t)} V(t_0^+, \boldsymbol{x}_0) \prod_{t_0<\tau_k<t} d_k, \quad t \geq t_0;$$

5.

$$g_0(t, w) = g(t, w) = -\frac{ae^{at}}{e^{at}} w = -aw, \quad a > 0$$

and $\psi_k(w) = d_k w,\ d_k \geq 0$ *for all* $k \in \mathbb{N},$ *then*

$$V(t, \boldsymbol{x}(t)) \leq \frac{e^{at_0}}{e^{at}} V(t_0^+, \boldsymbol{x}_0) \prod_{t_0<\tau_k<t} d_k, \quad t \geq t_0;$$

6. $g_0(t, w) = g(t, w) = -\alpha(w),\ \alpha \in \mathcal{K}$ *and* $\psi_k(w) = w$ *for all* $k \in \mathbb{N},$ *then*

$$V(t, \boldsymbol{x}(t)) \leq \varphi^{-1}[\varphi(V(t_0^+, \boldsymbol{x}_0)) - (t - t_0)], \quad t \geq t_0.$$

where $\dot{\varphi}(w) = 1/\alpha(w)$.

$\boxtimes$

We then present some special cases of Ξ' because their constructions are quite important to the proof of stability.

1. When $g_0(t, w) = 0$, then $v_{\max}(s, t_1, v_0) = v_0$. Therefore

$$\Xi' = \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s)) \leq V(t, \boldsymbol{x}(t)), s \in [t_0, t]\}.$$

2. When

$$g_0(t, w) = -\frac{\dot{\lambda}(t)}{\lambda(t)} w,$$

then

$$v_{\max}(s, t_1, v_0) = v_0 \frac{\lambda(t)}{\lambda(s)}, \quad s \in [t_0, t_1].$$

Therefore

$$\Xi' = \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s))\lambda(s) \leq V(t, \boldsymbol{x}(t))\lambda(t), s \in [t_0, t]\}.$$

3. When $g_0(t, w) = -\alpha(w)$, $\alpha \in \mathcal{K}$, then

$$v_{\max}(s, t_1, v_0) = \varphi^{-1}[\varphi(v_0) - (s - t_1)], \quad s \in [t_0, t_1]$$

where $\dot{\varphi}(w) = 1/\alpha(w)$. Because $v_{\max}(s, t_1, v_0)$ is increasing in s to the left of t_1, given an $s_0 < t_1$ and setting $\zeta(w) = v_{\max}(s_0, t_1, w)$, we find that $\zeta \in \mathcal{K}$. Therefore

$$\Xi' = \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s)) \leq \zeta(V(t, \boldsymbol{x}(t))), s \in [t_0, t]\}.$$

7.2.2 Stability in Terms of Two Measures

Theorem 7.2.2. *Let us assume that assumption 1 of Lemma 7.2.1 holds on each* $[\tau_k, \tau_{k+1}) \times \mathbb{R}_+$ *and*

1. $g_0 \in C[[\tau_k, \tau_{k+1}) \times \mathbb{R}_+, \mathbb{R}]$;
2. *let us define*

$$\Xi' \triangleq \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid V(s, \boldsymbol{x}(s)) \leq v_{\max}(s, t, V(t, \boldsymbol{x})), s \in [t_0, t]\}$$

   then $V(t, \boldsymbol{x}) \in \mathcal{V}_0$ *is locally Lipschitzian in* $\boldsymbol{x}$ *and for* $t \geq t_0$, $t \neq \tau_k$, *and* $\boldsymbol{x} \in \Xi'$ *we have*

$$D_- V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x}));$$

3. $V(t^+, \boldsymbol{x} + U(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x}))$ *for* $t = \tau_k$, $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ *is nonincreasing;*
4. $h_0, h \in \mathcal{H}$ *and* h_0 *is uniformly finer than* h;
5. $V(t, \boldsymbol{x})$ *is* h*-positive definite and* h_0*-decrescent;*
6. *there is a* $\rho_0 > 0$ *such that* $(t, \boldsymbol{x}) \in \mathcal{S}_{\rho_0}(h)$ *implies that* $(t, \boldsymbol{x} + U(k, \boldsymbol{x})) \in \mathcal{S}_\rho(h)$ *for all* $k \in \mathbb{N}$.

Then the stability properties of the trivial solution of the comparison system (7.47) imply the corresponding (h_0, h)*-stability properties of the impulsive control system (7.45).* ⊠

Proof.

1. Stability: Assume that the trivial solution of the comparison system (7.47) is stable. From the assumption that $V(t, \boldsymbol{x})$ is h-positive definite, we know that there are $\delta > 0$ and $\beta \in \mathcal{K}$ such that

$$h(t, \boldsymbol{x}) < \delta \Rightarrow \beta(h(t, \boldsymbol{x})) \leq V(t, \boldsymbol{x}). \tag{7.50}$$

   Given $0 < \eta < \delta_1 = \min(\delta, \rho_0, \rho)$ and $t_0 \in \mathbb{R}_+$, since the trivial solution of the comparison system (7.47) is stable, given $\beta(\eta) > 0$, there is a $\varepsilon_0 = \varepsilon_0(t_0, \eta) > 0$ such that for any solution, $w(t, t_0, w_0)$, of the comparison system (7.47)

$$w_0 \in [0, \varepsilon_0) \Rightarrow w(t, t_0, w_0) < \beta(\eta) \text{ for } t \geq t_0. \tag{7.51}$$

   Let us set $w_0 = V(t_0, \boldsymbol{x}_0)$, then from conditions 4 and 5, we know that there are a $\varepsilon_1 > 0$ and $\alpha \in \mathcal{K}$ such that

$$h(t_0, \boldsymbol{x}_0) < \delta, \quad V(t_0^+, \boldsymbol{x}_0) \leq \alpha(h_0(t_0, \boldsymbol{x}_0)), \quad (t_0, \boldsymbol{x}_0) \in \mathcal{S}_{\varepsilon_1}(h_0). \tag{7.52}$$

   Let us choose $\varepsilon = \varepsilon(t_0, \eta) \in (0, \varepsilon_1)$ such that $\alpha(\varepsilon) < \varepsilon_0$ and $h_0(t_0, \boldsymbol{x}_0) < \varepsilon$. We then have the following claim:
   Claim 7.2.2:

$$h_0(t_0, \boldsymbol{x}_0) < \varepsilon \Rightarrow h(t, \boldsymbol{x}(t)) < \eta, \quad t \geq t_0 \tag{7.53}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is any a solution of system (7.45).
If Claim 7.2.2 is false, then there are a solution, $\boldsymbol{x}_1(t) = \boldsymbol{x}_1(t, t_0, \boldsymbol{x}_0)$, of system (7.45) with $h_0(t_0, \boldsymbol{x}_0) < \varepsilon$ and a $t_1 \in (\tau_k, \tau_{k+1})$ for some k such that

$$h(t_1, \boldsymbol{x}(t_1)) \geq \eta \text{ and } h(t, \boldsymbol{x}(t)) < \eta \text{ for } t \in [t_0, \tau_k]. \tag{7.54}$$

From $\eta \in (0, \delta_1)$ we have $\eta \in (0, \rho_0)$. Then we have $h(\tau_k^+, \boldsymbol{x}(\tau_k) + U(k, \boldsymbol{x}(\tau_k))) < \rho$. Therefore there is a $t_2 \in (\tau_k, t_1]$ satisfying

$$\eta \leq h(t_2, \boldsymbol{x}(t_2)) < \rho. \tag{7.55}$$

It follows from Theorem 7.2.1 that

$$V(t, \boldsymbol{x}(t)) < w_{\max}(t, t_0, \alpha(h_0(t_0, \boldsymbol{x}_0))), \quad t \in [t_0, t_2] \tag{7.56}$$

Then we have the following contradiction:

$$\beta(\eta) \leq \beta(h(t_2, \boldsymbol{x}(t_2))) \leq V(t_2, \boldsymbol{x}(t_2)) < \beta(\eta). \tag{7.57}$$

Therefore, Claim 7.2.2 is true and the trivial solution of system (7.45) is (h_0, h)-stable.

2. Uniform stability: Assume that the trivial solution of the comparison system (7.47) is uniformly stable . Then ε is independent of t_0. Therefore, the trivial solution of system (7.45) is (h_0, h)-uniformly stable.
3. Asymptotic stability: Assume that the trivial solution of the comparison system (7.47) is asymptotically stable . Then from the first part of this proof we know that the trivial solution of system (7.45) is (h_0, h)-stable. Let us set $\eta = \delta_1$ and $\varepsilon_2 = \varepsilon(t_0, \delta_1)$, we then have

$$h_0(t_0, \boldsymbol{x}_0)) < \varepsilon_2 \Rightarrow h(t, \boldsymbol{x}(t)) < \rho \text{ for } t \geq t_0. \tag{7.58}$$

Given $\eta \in (0, \delta_1)$, $\beta(\eta) > 0$ and $t_0 \in \mathbb{R}_+$, from the fact that the trivial solution of the comparison system (7.47) is attractive, we know that there are a $\varepsilon_3 = \varepsilon_3(t_0) > 0$ and a $T = T(t_0, \eta) > 0$ such that

$$w_0 \in [0, \varepsilon_3) \Rightarrow w(t, t_0, w_0) < \beta(\eta) \text{ for } t \geq t_0 + T. \tag{7.59}$$

Let us choose $\varepsilon_0 = \min(\varepsilon_2, \varepsilon_3)$ and let $h_0(t_0, \boldsymbol{x}_0) < \varepsilon_0$, then from (7.58) and the arguments leading to (7.56) we have

$$V(t, \boldsymbol{x}(t)) \leq w_{\max}(t, t_0, \alpha(h_0(t_0, \boldsymbol{x}_0))) \text{ for } t \geq t_0 \tag{7.60}$$

from which we have

$$\begin{aligned} \beta(h(t, \boldsymbol{x}(t))) &\leq V(t, \boldsymbol{x}(t)) \\ &\leq w_{\max}(t, t_0, \alpha(h_0(t_0, \boldsymbol{x}_0))) \\ &< \beta(\eta), \quad t \geq t_0 + T. \end{aligned} \tag{7.61}$$

Therefore we know that $h(t, \boldsymbol{x}(t)) < \eta$ for $t \geq t_0 + T$, which proves that the trivial solution of system (7.45) is attractive. This means that the trivial solution of system (7.45) is (h_0, h)-asymptotically stable.

4. Uniform asymptotic stability: Assume that the trivial solution of the comparison system (7.47) is uniformly asymptotically stable . Then ε_0 and T are independent of t_0. Therefore, the trivial solution of system (7.45) is (h_0, h)-uniformly asymptotically stable.

■

From Theorem 7.2.2 we have the following corollaries that are convenient to use in designing impulsive controllers.

Corollary 7.2.2. *In Theorem 7.2.2, let us assume that* $g_0(t, w) = g(t, w) = 0$, $\psi_k(w) = d_k w$, $d_k \geq 0$ *for all* k, *then the trivial solution of system (7.45) is* (h_0, h)*-uniformly stable if*

$$\prod_{k=1}^{\infty} d_k < \infty.$$

⊠

Proof. The proof is immediately followed from Corollary 7.2.1. ■

Corollary 7.2.3. *In Theorem 7.2.2, let us assume that* $g_0(t, w) = 0$, $g(t, w) = \dot{\lambda}(t)w$, $\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$, $\dot{\lambda}(t) \geq 0$, $\psi_k(w) = d_k w$, $d_k \geq 0$ *for all* k, *then the trivial solution of system (7.45) is*

1. (h_0, h)*-stable if*

$$\lambda(\tau_k) + \ln d_k \leq \lambda(\tau_{k-1}) \text{ for all } k; \tag{7.62}$$

2. (h_0, h)*-asymptotically stable if there is a* $\gamma > 1$ *such that*

$$\lambda(\tau_k) + \ln(\gamma d_k) \leq \lambda(\tau_{k-1}) \text{ for all } k. \tag{7.63}$$

⊠

Proof. The comparison system (7.47) becomes

$$\begin{cases} \dot{w} = \dot{\lambda}(t)w, & t \neq \tau_k, \\ w(\tau_k^+) = d_k w(\tau_k), & t = \tau_k, \\ w(t_0^+) = w_0 \geq 0. \end{cases} \tag{7.64}$$

The solution of (7.64) for $t \geq t_0$ is given by

$$w(t, t_0, w_0) = w_0 \prod_{t_0 < \tau_k < t} d_k e^{\lambda(t) - \lambda(t_0)}. \tag{7.65}$$

Without loss of generality, let us suppose that $t_0 \in (0, \tau_1)$. Because $\lambda(t)$ is nondecreasing, from (7.62) we have for $t \geq t_0$

$$w(t, t_0, w_0) \leq w_0 e^{\lambda(\tau_1) - \lambda(t_0)}. \tag{7.66}$$

Let us choose

$$\varepsilon = \frac{\eta}{2} e^{\lambda(\tau_0)-\lambda(t_1)}$$

then we can prove the trivial solution of (7.64) is stable. This proves conclusion 1.

From (7.63) it follows that for $t \geq t_0$

$$w(t, t_0, w_0) \leq \frac{1}{\gamma^k} w_0 e^{\lambda(\tau_1)-\lambda(t_0)}, \quad t \in (\tau_{k-1}, \tau_k] \tag{7.67}$$

from which we have

$$\lim_{t\to\infty} w(t, t_0, w_0) = 0. \tag{7.68}$$

This proves conclusion 2. ∎

Then let us consider the following impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \int_{t_0}^{t} \boldsymbol{u}(t, s, \boldsymbol{x}(s))ds, & t \neq \tau_k, \\ \Delta\boldsymbol{x} = U(k, \boldsymbol{x}(\tau_k)), & t = \tau_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0, & t_0 \geq 0, \quad k \in \mathbb{N} \end{cases}} \tag{7.69}$$

which is a special case of system (7.45). Let us suppose that

$$\|\boldsymbol{u}(t, s, \boldsymbol{x})\| \leq \gamma(t, s)\|\boldsymbol{x}\| \text{ on } \mathbb{R}_+ \times \mathcal{S}_\rho$$

and other assumptions are the same as those for system (7.45).

Let us construct a Lyapunov function as $V(t, \boldsymbol{x}) = \|\boldsymbol{x}\|$ and choose the set

$$\Xi' = \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathcal{S}_\rho] \mid \|\boldsymbol{x}(s)\| \leq \|\boldsymbol{x}(t)\|, \quad s \in [t_0, t]\}.$$

Then we have

$$D_-V(t, \boldsymbol{x}) \leq \left(\mu(A) + \int_{t_0}^{t} \gamma(t, s)ds\right) V(t, \boldsymbol{x})$$

where $\mu(A)$ is the logarithmic norm of A given by

$$\mu(A) \triangleq \lim_{h\to 0} \frac{\|I + hA\| - 1}{h}.$$

Therefore we have $g_0(t, w) = 0$ and $g(t, w) = \dot{\lambda}(t)w$ where

$$\dot{\lambda}(t) = \mu(A) + \int_{t_0}^{t} \gamma(t, s)ds.$$

Then we can choose different impulsive control laws $U(k, \boldsymbol{x})$ to make the trivial solution of comparison system (7.64) stable. For example, we can choose a kind of $U(k, \boldsymbol{x})$ such that $\psi_k(w) = d_k w$, $d_k \geq 0$ for all $k \in \mathbb{N}$, then the stability properties can be found by using Corollary 7.2.3.

7.2.3 Practical Stability

Based on the comparison theorem (Theorem 7.2.1) and the similar technology presented in Section 6.1, we can prove the following theorem[15] concerning the practical stability of system (7.45).

Theorem 7.2.3. *Let $0 < \mu < \nu$ be given and there is such a $\rho = \rho(\nu) > 0$ that if $\boldsymbol{x} \in \mathcal{S}_\nu$ then $\boldsymbol{x} + U(k, \boldsymbol{x}) \in \mathcal{S}_\rho$ for each $k \in \mathbb{N}$. Let us assume that there are $\alpha, \beta \in \mathcal{K}$ such that $\alpha(\mu) < \beta(\nu)$ and*

$$\beta(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \alpha(\|\boldsymbol{x}\|) \textit{ for } (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathcal{S}_\rho.$$

Then the practical stability properties of the comparison system (7.47) imply the corresponding practical stability properties of control system (7.45). ⊠

As an example, let us study the practical stability of system (7.69). Let us suppose that

$$\|\boldsymbol{u}(t, s, \boldsymbol{x})\| \leq \gamma(t, s)\|\boldsymbol{x}\| \text{ on } \mathbb{R}_+ \times \mathbb{R}^n$$

and other assumptions are the same as those for system (7.45).

Let us construct a Lyapunov function as $V(t, \boldsymbol{x}) = \|\boldsymbol{x}\|e^{\varpi t}$, $\varpi > 0$ and choose the set

$$\Xi' = \{\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}_+, \mathbb{R}^n] \mid \|\boldsymbol{x}(s)\|e^{\varpi s} \leq \|\boldsymbol{x}(t)\|e^{\varpi t}, \quad s \in [t_0, t]\}.$$

Then we have

$$D_- V(t, \boldsymbol{x}) \leq \left(\varpi + \mu(A) + \int_{t_0}^{t} \gamma(t, s)e^{\varpi(t-s)} ds\right) V(t, \boldsymbol{x})$$

where $\mu(A)$ is the logarithmic norm of A. Therefore we have $g(t, w) = \dot{\lambda}(t)w$ where

$$\dot{\lambda}(t) = \varpi + \mu(A) + \int_{t_0}^{t} \gamma(t, s)e^{\varpi(t-s)} ds.$$

Then we can choose different impulsive control laws $U(k, \boldsymbol{x})$ to make the trivial solution of comparison system (7.64) stable. For example, we can choose a kind of $U(k, \boldsymbol{x})$ such that $\psi_k(w) = d_k w$, $d_k \geq 0$ for all $k \in \mathbb{N}$, then the stability properties can be found by using Corollary 6.1.2 and Theorem 7.2.3.

Note 7.2.1. Section 7.1 is adopted from [28]. Section 7.2 is adopted from [17].

♦

8. Impulsive Computational Verb Control

Computational verbs and computational verb systems[31, 32, 35, 34, 33, 36, 38, 40, 39, 42, 41, 43] are revolutionary paradigms working together with fuzzy systems[54, 55, 56] for the purpose of embedding experts' knowledge that coded in human natural languages into machine intelligence. In this chapter we present impulsive control strategies based on newly developed computational verb control systems. Control with computational verb, or *verb control* is a systematic way to integrate human experts' knowledge of dynamical processes into control systems. Since so far there is no reference addressing verb control systems, in this chapter we shall first present verb control system without impulsive control strategy to give the reader the flavor of verb control systems. Since the basic knowledge of computational verb systems had been well-established in the references listed above and especially in [43], we do not repeat the definitions and examples of computational verbs in this chapter.

The basic structure of a verb control system is shown in Fig. 8.1. Observe that a typical verb control system consists of a verb recognition block(*verbification*), a set of verb control rules and a *verb collapse* block(*deverbification*). The verb recognition block is used to verbify a dynamical output of the plant. The outputs of a verb recognition block are called *observing verbs*. The verb inference engine and verb rule base are used to choose controlling verbs with respect to the reference signal which is used to shift the equilibrium points of the controlled system. The verb collapse block(deverbification) is used to transform the controlling verbs into control signals.

The design of verb control system may include: the definition of input and output variables, the selection of data manipulation method, the design of outer systems of computational verbs and the verb control rule design. The source of knowledge to construct the verb control rules is the control experiences of human experts. The control experience consists of a set of conditional "IF-THEN" statements, where the IF-part contains conditions modeled by observing verbs and the THEN-part provides actions expressed by controlling verbs.

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 219−235, 2001.

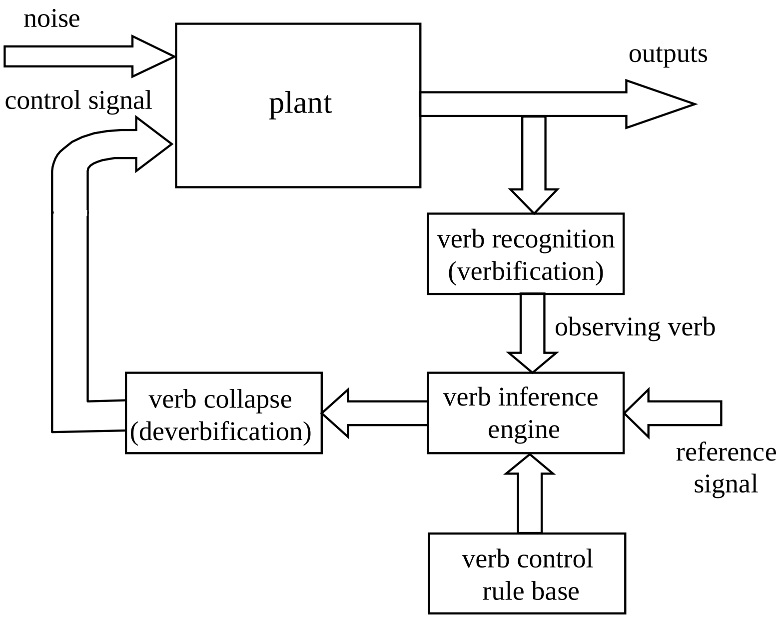


Fig. 8.1. The block diagram of a typical verb control system.

8.1 Design of Verb Controller with Fuzzy Errors

In a typical verb controller, we usually know error $e(t)$ between the output of the plant and the reference signal and changes in error $r(t) \triangleq \dot{e}(t)$. The control input $u(t)$ is generated by the deverbification block. Without loss of generality, let us assume that $e(t)$, $r(t)$ and $u(t)$ are normalized into interval $[-1, 1]$. Thus, the associative universe of discourse of $e(t)$, $r(t)$ and $u(t)$ confined within $[-1, 1]$.

The structure of verb controller with respect to $e(t)$, $r(t)$ and $u(t)$ is shown in Fig. 8.2. Observe that a verb controller consists of three parts: a "verbification" block, a "verb inference engine" and a "deverbification" block. The details of each block will be addressed in this section.

Verbifications Block

In the verbification block the observed waveforms of $e(\tau)$ and $r(\tau)$ in a time period $\tau \in [t - \Delta, t]$ are used to determine which verb can be used to model the dynamics of the control error at time moment t. For simplicity, we use $e(t - \Delta, t)$ and $r(t - \Delta, t)$ to denote the waveforms of $e(\tau)$ and $r(\tau)$ with $\tau \in [t - \Delta, t]$. Then the universe of discourse of $e(t - \Delta, t)$ and $r(t - \Delta, t)$ are respectively given by

$$U_e(\Delta) = [t - \Delta, t] \times [-1, 1], \quad U_r(\Delta) = [t - \Delta, t] \times [-1, 1].$$

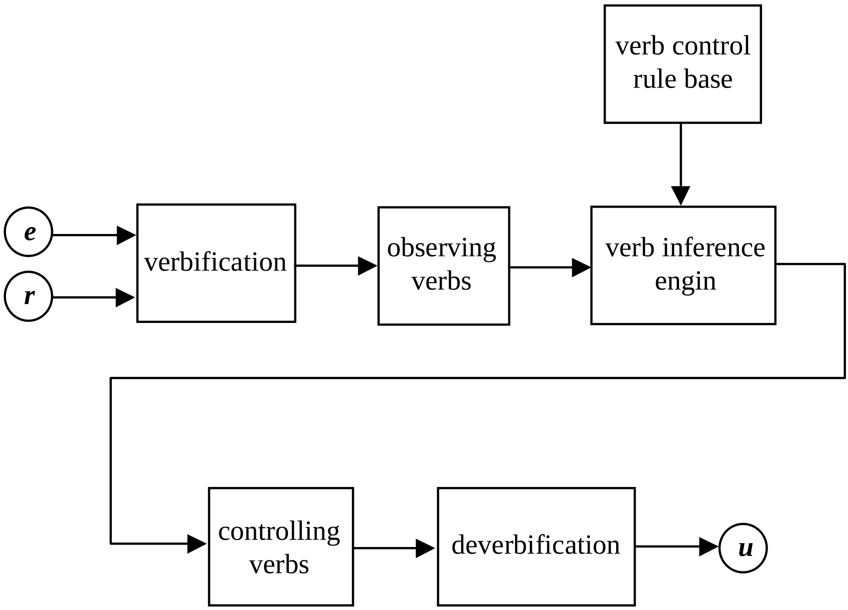


Fig. 8.2. The block diagram of a typical verb controller.

To reduce redundant information in these values, let us partition the values of $e(t) \in [-1, 1]$ and $r(t) \in [-1, 1]$ into two sets of fuzzy values T_e and T_r, respectively. We also suppose both term-sets T_e and T_r have the same number of linguistic members on both positive and negative sides

$$\begin{aligned} T_e &= \{E_{-K}, \cdots, E_{-1}, E_0, E_1, \cdots, E_K\} \\ T_r &= \{R_{-K}, \cdots, R_{-1}, R_0, R_1, \cdots, R_K\}. \end{aligned} \tag{8.1}$$

Therefore, the universe of discourse of $e(t)$ is partitioned into $2K+1$ sections and each section is modeled with a fuzzy value $E_i, i = -K, \cdots, 0, \cdots, K$ that are characterized by membership functions $\mu_{E_i}(e(t))$. Similarly, the universe of discourse of $r(t)$ is also partitioned into $2K+1$ sections and each section is modeled by a fuzzy value $R_i, i = -K, \cdots, 0, \cdots, K$ that are characterized by membership functions $\mu_{R_i}(e(t))$.

Let $V_{\text{verbification}}$ denote the set of computational verbs used to model the dynamics of fuzzy values $\mu_{E_i}(e(t-\Delta, t]))$ and $\mu_{R_i}(e(t-\Delta, t])), i = -K, \cdots, 0, \cdots, K$. Then we can use the following steps to verbify the control errors:

1. Decompose any verb $\mathcal{V}_j \in V_{\text{verbification}}$ into verb rules described by simple verbs. For example, suppose fluctuate is a verb in $V_{\text{verbification}}$, we need to use the following relations to define three possible decompositions:

$$\begin{aligned}
&\mathsf{fluctuate}\\
&\triangleq \text{a center verb}\\
&\triangleq \text{a center verb}\\
&\quad +[(\text{a node verb}) \circ (\text{adverbial defining small value change})]\\
&\triangleq (\text{a focus verb}) \circ (\text{adverbial defining value range}). \qquad (8.2)
\end{aligned}$$

2. Use similarity criteria to determine which verb in $V_{\text{verbification}}$ can be use to model the dynamics of control error. This step needs to run verb similarity test for each possible decomposition of every element in $V_{\text{verbification}}$.

Verb Inference Engine

In this block, we use verb reasoning to implement a verb inference engine for choosing different controlling verbs. Let V_{control} denote the set of controlling verbs, and let $\mathcal{V}_i^o \in V_{\text{verbification}}$ and $\mathcal{V}_j^c \in V_{\text{control}}$ we then have the verb inference rules as

$$\begin{aligned}
&\text{IF the control error } \mathcal{V}_1^o, \text{ THEN the control } \mathcal{V}_1^c;\\
&\text{IF the control error } \mathcal{V}_2^o, \text{ THEN the control } \mathcal{V}_2^c;\\
&\qquad\vdots\\
&\text{IF the control error } \mathcal{V}_M^o, \text{ THEN the control } \mathcal{V}_M^c. \qquad (8.3)
\end{aligned}$$

Observe that there are M verb IF-THEN rules in this verb inference engine. In this set of verb rules, $\mathcal{V}_i^o, i = 1, 2, \cdots, M$ are not necessary to be M distinguished verbs. Similarly, $\mathcal{V}_i^c, i = 1, 2, \cdots, M$ are also not necessary to be M distinguished verbs. The output of this block is a controlling verb that can be used to determine the value of control signal in the deverbification block.

Deverbification Block

In this block controlling verbs collapse into a control signals that used to control the plant. The definition and construction of collapses of verbs can be found in [43, 41].

8.2 Design of Verb Controller with Verb Singletons

In this section we present a design example with the lifetime of each verb as $\Delta = 0$. In this case, the verbs become *verb singletons* which are described by a sample of the outer system and a sample of the changing rate of the outer system. Let us suppose that the verb controller is implemented by a digital computer with a sampling interval δ, then we design different blocks of a verb controller as follows.

Normalization

Assume that the range of the measured control error $\tilde{e}(n\delta) \triangleq \tilde{e}(n)$ is $[-1/G_E, -1/G_E]$, then the normalized control error $e(n) = G_E\tilde{e}(n) \in [-1, 1]$. In a digital implementation, we define the change of control error as $\tilde{r}(n) = \tilde{e}(n) - \tilde{e}(n-1)$. Assume that the range of $\tilde{r}(n)$ is $[-1/G_R, -1/G_R]$, then we normalize $\tilde{r}(n)$ into $r(n) = G_R\tilde{r}(n) \in [-1, 1]$.

Verbification

The inputs of the verification block are e and r. The output of this block is an observing verb. Let us partition the universes of discourse of e and r into $2K + 1$ adjectives for each; namely,

$$\begin{aligned} T_e &= \{E_{-K}, \cdots, E_{-1}, E_0, E_1, \cdots, E_K\} \\ T_r &= \{R_{-K}, \cdots, R_{-1}, R_0, R_1, \cdots, R_K\} \end{aligned} \tag{8.4}$$

where E_i and R_i are adjectives for describing the values of e and r, respectively. Then the verbification is given by the following set of rules called *verbification rules*:

$$R_{i,j}: \text{ IF } e_n \text{ is } E_i \text{ AND } r_n \text{ is } R_j, \text{ THEN the control error } \mathcal{V}^v_{i+j}. \tag{8.5}$$

Observe that we need $4K + 2$ verbs to model the control error and the verbification rule is symmetric with respect to e and r because $\mathcal{V}^v_{i+j} = \mathcal{V}^v_{j+i}$. This design is just for the convenience of analysis. We then use the membership function of each adjective to calculate the following membership values for $e(n)$ and $r(n)$

$$\begin{aligned} &E_i(e(n)) = \mu_{E_i}(e(n)) \in [0, 1], \qquad R_i(r(n)) = \mu_{R_i}(r(n)) \in [0, 1], \\ &i = -K, \cdots, 0, \cdots, K. \end{aligned} \tag{8.6}$$

We then calculate the *firing level* of verb $\mathcal{V}^v_{i+j}$, $\phi_{i,j}$ as

$$\phi_{i,j} = \mathbb{F}(E_i(e_n), R_j(r_n)) \tag{8.7}$$

where $\mathbb{F} : [0, 1] \times [0, 1] \to [0, 1]$ can be any function for implementing a verb similarity judgement. For example, we can choose $\mathbb{F}$ as

$$\mathbb{F}(x, y) = \min(x, y). \tag{8.8}$$

We then need to calculate the final choice of the observing verb as

$$\mathcal{V}_o = \frac{\sum_{i=-K}^{K}\sum_{j=-K}^{K} \mathbb{W}(\phi_{i,j}, \mathcal{V}^v_{i+j})}{\sum_{i=-K}^{K}\sum_{j=-K}^{K} \phi_{i,j}} \tag{8.9}$$

where $\mathbb{W}$ is a weighting operator that impose an *adverbial* upon the verb $\mathcal{V}^v_{i+j}$. Observe that the observing verb $\mathcal{V}_o$ is a compound verb that generated by $4K+2$ verbs $\mathcal{V}^v_{i+j}$. For designing simple verb controller, we can choose $\mathbb{W}$ as simple abverbial. For example, we can use the following simple operation:

$$\mathcal{F}_{\mathcal{V}_o} = \frac{\sum_{i=-K}^{K}\sum_{j=-K}^{K}\phi_{i,j}\mathcal{F}_{\mathcal{V}^v_{i+j}}}{\sum_{i=-K}^{K}\sum_{j=-K}^{K}\phi_{i,j}} \tag{8.10}$$

which means that the outer system of the observing verb is the center of gravity of activated verbs $\mathcal{V}^v_{i+j}$.

Verb Inference Engine

Verb inference is based on a set of verb rules that connect observing verbs $\mathcal{V}^o_k$ to controlling verbs $\mathcal{V}^c_k$ by using rule set (8.3). The output of this block is a compound verb given by

$$\mathcal{V}_u = \frac{\sum_{k=1}^{4K+2}\mathbb{A}(\mathbb{S}(\mathcal{V}^o_k,\mathcal{V}_o),\mathcal{V}^c_k)}{\sum_{k=1}^{4K+2}\mathbb{S}(\mathcal{V}^o_k,\mathcal{V}_o)} \tag{8.11}$$

where $\mathbb{S}$ is an operation to determine the similarity between two verbs and $\mathbb{A}$ is an operation to determine the contribution of each verb rule to the final choice of controlling verb $\mathcal{V}_u$.

Deverbification

This block outputs a control signal based on $\mathcal{V}_u$. It functions as a collapse map.

8.3 Examples of Verb Control Systems

In this section, I present examples of verb control of a chaotic circuit called Chua's circuit[5] that consists of two linear capacitors C_1 and C_2, a linear inductor L, two linear resistors R and R_0, and a piecewise-linear negative resistor. This chaotic circuit is described by the following state equation:

$$\begin{cases} \frac{dv_1}{dt} = \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} = \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} = -\frac{1}{L}[v_2 + R_0 i_3] \end{cases} \tag{8.12}$$

where $G = 1/R$ and $f(\cdot)$ is the piecewise-linear characteristic of the negative resistor defined by

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)(|v_1 + E| - |v_1 - E|) \tag{8.13}$$

where E is the breakpoint voltage. The output of this chaotic system is given by

$$y(t) = v_1(t) \tag{8.14}$$

Since the structures of verb controllers for controlling this chaotic circuit are very simple, we do not need to design them block by block. I will present some design examples.

Example 8.3.1. In this example, the goal of verb control is to keep the chaotic system staying in the range of $y > 0$. To do this, a human expert can intuitively (and immediately) find the following verb rule:

if $y(t)$ goes across zero from positive side, then capture it back.

The control signal can be written as

$$u = \mathcal{F}_{\mathsf{capture}} \tag{8.15}$$

We choose $\mathcal{F}_{\mathsf{capture}}$ as

$$\mathcal{F}_{\mathsf{capture}} = \begin{cases} 0, & \text{if } r(t) < h \\ -0.01(y(t) - 0.5)/C_1, & \text{else} \end{cases} \tag{8.16}$$

where h is a threshold, $r(t)$ is given by

$$r(t) = \frac{1}{T}\int_0^T |y(t-\tau) - \mathcal{F}_{\mathsf{go}}(T-\tau)|d\tau \tag{8.17}$$

where T is the window length of the outer system of go. $\mathcal{F}_{\mathsf{go}}$ is given by observing the uncontrolled chaotic system.

The verb-controlled chaotic system is given by

$$\begin{cases} \frac{dv_1}{dt} = \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] + u \\ \frac{dv_2}{dt} = \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} = -\frac{1}{L}[v_2 + R_0 i_3] \end{cases}$$
$$u = \mathcal{F}_{\mathsf{capture}} = \begin{cases} 0, & \text{if } r(t) < h \\ -0.01(y(t) - 0.5)/C_1, & \text{else} \end{cases} \tag{8.18}$$

The simulation result is shown in Fig. 8.3. The parameters for this simulation is as follows: $C_1 = 5.56nF$, $C_2 = 50nF$, $G = 0.7mS$, $L = 7.14mH$, $G_a = -0.81mS$, $G_b = -0.5mS$, $E = 1$, $R_0 = 0$, $h = 1.189918$, and $T = 0.3ms$. The fourth order Runge-Kutta method with fixed step-size of $1\mu s$ is used. Figure 8.3(a) shows the uncontrolled waveform of $v_1(t)$. Observe that almost half of the time, the chaotic system enters the region of $v_1(t) < 0$. After observing this uncontrolled waveform, we find that before

$y(t)$ goes across zero from the positive direction, there exists a typical unstable oscillation. This typical observation is then stored as a waveform of the outer system of go. Figure 8.3(b) shows the observed outer system of go (across 0). This waveform can be stored into a computer by a few space yet it is not necessary to know the underlying model of this waveform, which is usually done by other control strategies such as model identification. Figure 8.3(c) shows the controlled waveform of $v_1(t)$. Observe that there still exists some negative peaks due to the transient of the verb feedback control loop. Figure 8.3(d) shows $r(t)$, which is used to construct $\mathcal{F}_{\mathsf{capture}}$.

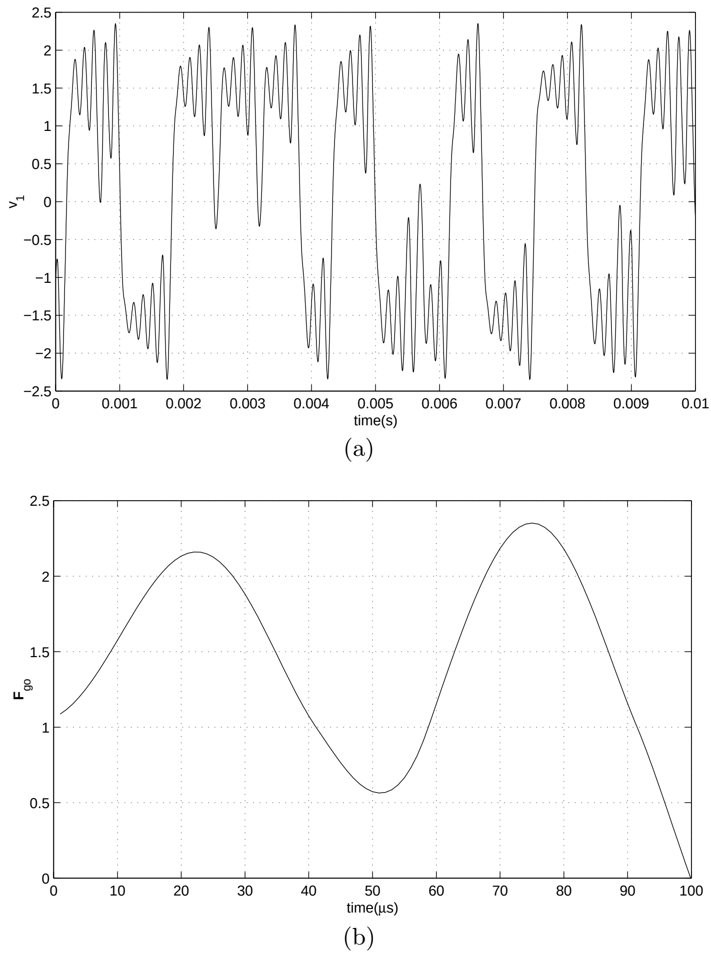
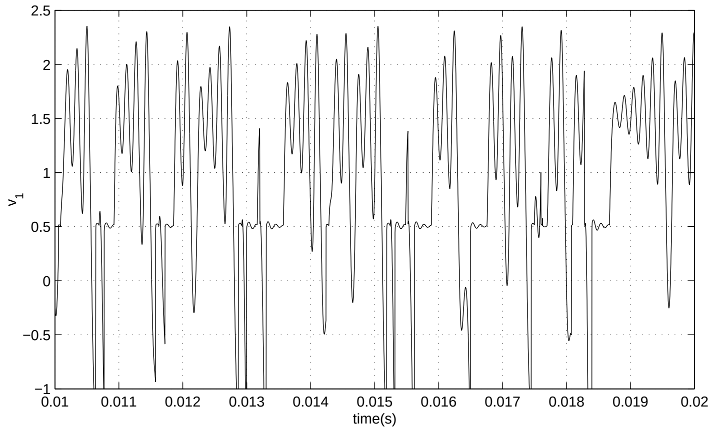
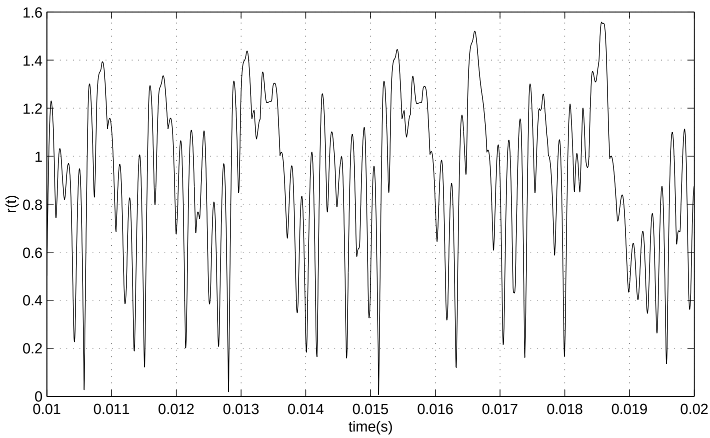


Fig. 8.3. Verb control of chaotic circuit to a desired region. (a) The waveform of $v_1(t)$ of the free chaotic system. (b) The computational verb go observed from the free waveform of $v_1(t)$. (c) The controlled waveform of $v_1(t)$. (d) $r(t)$.



(c)



(d)

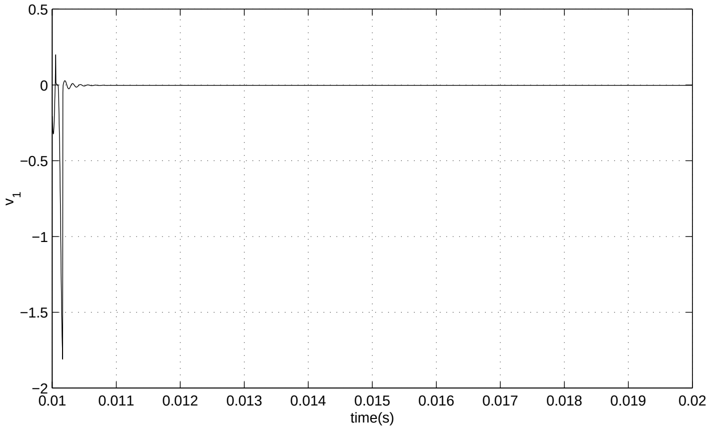
Fig. 8.3 (*Continued*).

★

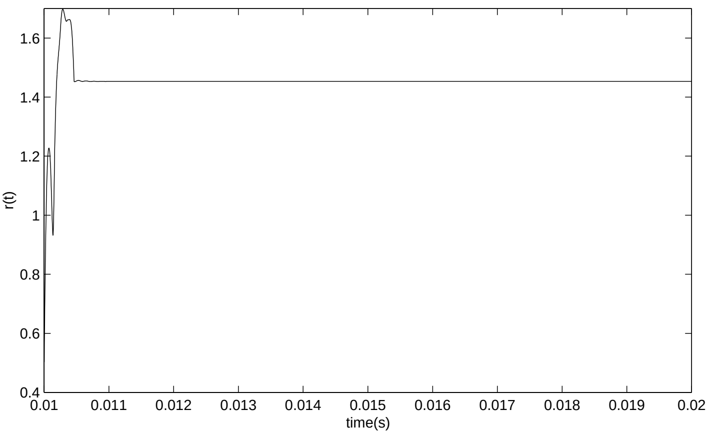
Example 8.3.2. In this example, we control the chaotic system to a fixed point. We can use different control laws. For example the following control law, which control the chaotic system to a fixed point, is constructed by changing the second part of (8.18) into

$$u = \mathcal{F}_{\mathsf{capture}} = \begin{cases} 0, & \text{if } r(t) < h \\ -0.01[y(t) - 0.01(r(t) - h)]/C_1, & \text{else} \end{cases} \tag{8.19}$$

with $h = 1.189918$. The simulation results are shown in Fig.8.4. Figure 8.4(a) shows the controlled waveform, which approaches to -0.002663. Figure 8.4(b) shows the corresponding $r(t)$, which approaches to 1.453270. All other conditions are kept the same as those in Example 8.3.1. ★



(a)



(b)

Fig. 8.4. Verb control of chaotic circuit to a fixed point. (a) The controlled waveform of $v_1(t)$. (b) $r(t)$.

8.4 Linear Verb Control Systems

We define a *linear verb control system* as a verb control system where all observing verbs and controlling verbs are modeled by linear systems. In this case, the evolving systems of observing verbs and controlling verbs are sets of linear ordinary differential equations or linear difference equations. We also assume a simple structure such that the inner system of all observing verbs is the same and only let the choice of different measurements happens in the outer systems of observing verbs.

8.4.1 Using Different Controlling Verbs

The verb control rules are given by

IF the plant $\mathcal{V}_1^p$, THEN the controller $\mathcal{V}_1^c$; IF the plant $\mathcal{V}_2^p$, THEN the controller $\mathcal{V}_2^c$; $\vdots$ IF the plant $\mathcal{V}_\nu^p$, THEN the controller $\mathcal{V}_\nu^c$; ELSE the controller $\mathcal{V}_0^c$.

where $\mathcal{V}_i^p, i = 1, 2, ..., \nu$, are verb statements constructed by observing verbs. $\mathcal{V}_i^c, i = 0, 1, ..., \nu$, are verb statements constructed by controlling verbs. Let the plant be

$$\begin{cases} \dot{\boldsymbol{x}}_p = A\boldsymbol{x}_p + B\boldsymbol{u}, \\ \boldsymbol{y} = C\boldsymbol{x}_p, \qquad \leftarrow \text{outer system of } \mathcal{V}_i^p, i = 1, 2, ..., \nu \end{cases} \tag{8.20}$$

where $\boldsymbol{x}_p \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{y} \in \mathbb{R}^l$ are state variables, control input and output, respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$ are constant matrices. Notice that all observing verbs have the same inner system and outer system. However, there will be different collapse maps for different observing verbs when verb logic is used to determined which controlling verb is activated.

Let the evolving system of the i-th controlling verb be

$$\begin{cases} \dot{\boldsymbol{x}}_i = A_i\boldsymbol{x}_i + B_i\boldsymbol{u}, \\ \boldsymbol{u} = C_i\boldsymbol{x}_i + D_i\boldsymbol{y}, \leftarrow \text{outer system of } \mathcal{V}_i^c, i = 0, 1, ..., \nu. \end{cases} \tag{8.21}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ is the state variable of $\mathcal{V}_i$. $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m}$, $C_i \in \mathbb{R}^{m\times n_i}$ and $D_i \in \mathbb{R}^{m\times l}$ are constant matrices.

When the controller is governed by the i-th controlling verb $\mathcal{V}_i^c$, the closed loop system is given by

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{x}}_p \\ \dot{\boldsymbol{x}}_i \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \begin{bmatrix} A + BD_iC & BC_i \\ B_iC & A_i \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{x}_i \end{bmatrix} = (K + \delta K_i) \underbrace{\begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{x}_i \end{bmatrix}}_{\boldsymbol{x}} \tag{8.22}$$

where $K \in \mathbb{R}^{(n+n_1)\times(n+n_1)}$ and $\delta K_i \in \mathbb{R}^{(n+n_1)\times(n+n_1)}$ can be viewed as the standard parameter matrix and the perturbing parameter matrix for the verb control system. Suppose that all controlling verbs share K as their common part and K is asymptotically stable, then the stability of the entire verb control system is guaranteed by the following theorem.

Theorem 8.4.1. *Let P be the unique positive definite symmetric solution of*

$$K^\top P + PK + Q = 0 \tag{8.23}$$

where Q is a positive definite symmetric matrix and K is asymptotically stable. If all controlling verbs satisfy the following condition

$$\|\delta K^\top P + P\delta K\|_2 \le \lambda_{\min}(Q) \tag{8.24}$$

where δK is any of δK_i, then the linear verb control system is stable. ⊠

Proof. Followed Lyapunov's Theorem we know if K is asymptotically stable, we can construct the following Lyapunov function:

$$V(\boldsymbol{x}) = \boldsymbol{x}^\top P\boldsymbol{x} \tag{8.25}$$

with $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq 0$ and $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$. Take the derivative of $V(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ along solutions of (8.22) we have

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \dot{\boldsymbol{x}}^\top P\boldsymbol{x} + \boldsymbol{x}^\top P\dot{\boldsymbol{x}}\\ &= \boldsymbol{x}^\top(K^\top + \delta K^\top)P\boldsymbol{x} + \boldsymbol{x}^\top P(K + \delta K)\boldsymbol{x}\\ &= \boldsymbol{x}^\top(K^\top P + PK)\boldsymbol{x} + \boldsymbol{x}^\top(\delta K^\top P + P\delta K)\boldsymbol{x}\\ &= -\boldsymbol{x}^\top Q\boldsymbol{x} + \boldsymbol{x}^\top(\delta K^\top P + P\delta K)\boldsymbol{x} \qquad \Leftarrow (8.23).\end{aligned} \tag{8.26}$$

From condition

$$\|\delta K^\top P + P\delta K\|_2 \le \lambda_{\min}(Q) \tag{8.27}$$

and the fact

$$\begin{aligned}|\boldsymbol{x}^\top(\delta K^\top P + P\delta K)\boldsymbol{x}| &\le \|\boldsymbol{x}^\top\|_2\|\delta K^\top P + P\delta K\|_2\|\boldsymbol{x}\|_2\\ &\le \|\boldsymbol{x}\|_2^2\|\delta K^\top P + P\delta K\|_2\end{aligned} \tag{8.28}$$

we have

$$\boldsymbol{x}^\top(\delta K^\top P + P\delta K)\boldsymbol{x} \le \lambda_{\min}(Q)\boldsymbol{x}^\top\boldsymbol{x}. \tag{8.29}$$

From Rayleigh principle we have

$$\boldsymbol{x}^\top Q\boldsymbol{x} \ge \lambda_{\min}(Q)\boldsymbol{x}^\top\boldsymbol{x} \tag{8.30}$$

Thus from (8.29) and (8.30) we have

$$\boldsymbol{x}^\top(\delta K^\top P + P\delta K)\boldsymbol{x} \le \boldsymbol{x}^\top Q\boldsymbol{x} \tag{8.31}$$

then from (8.31) and (8.26) we immediate have $\dot{V}(\boldsymbol{x}) \le 0$. ∎

Remark 8.4.1. This theorem tells us that if the parameter perturbations introduced by every controlling verb satisfy condition (8.24), then the entire verb controlled system is asymptotically stable. ♦

8.4.2 Using Single Controlling Verb

In this section we discuss a kind of verb control rule whose controlling verbs are the same with different adverbs to give different constrains. This kind of verb control rule is given by

IF the plant $\mathcal{V}_1^p$, THEN the controller $\mathcal{V}^c$ $\texttt{adverb}_1$; IF the plant $\mathcal{V}_2^p$, THEN the controller $\mathcal{V}^c$ $\texttt{adverb}_2$; $\vdots$ IF the plant $\mathcal{V}_n^p$, THEN the controller $\mathcal{V}^c$ $\texttt{adverb}_n$; ELSE the controller $\mathcal{V}^c$ $\texttt{adverb}_0$.

In this case, the structure of the controlling verb $\mathcal{V}^c$ is the same for THEN-rules and the ELSE-rule. We usually use the adverbs to change the parameters of the controlling verb as those modeled in [33] as follows:

$$A_i = \tilde{A} + \delta A_i, B_i = \tilde{B} + \delta B_i, C_i = \tilde{C} + \delta C_i, D_i = \tilde{D} + \delta D_i. \tag{8.32}$$

Then the verb controlled system is given by

$$\begin{bmatrix} \dot{\boldsymbol{x}}_p \\ \dot{\boldsymbol{x}}_i \end{bmatrix} = \left(\underbrace{\begin{bmatrix} A + B\tilde{D}C & B\tilde{C} \\ \tilde{B}C & \tilde{A} \end{bmatrix}}_{K} + \underbrace{\begin{bmatrix} B\delta D_i C & B\delta C_i \\ \delta B_i C & \delta A_i \end{bmatrix}}_{\delta K_i} \right) \begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{x}_i \end{bmatrix} \tag{8.33}$$

We would like to find the range within which an adverb can change the controlling verb such that the entire control system still in stable region. First, let $p_j, j = 1, 2, ..., m$, denote all tunable parameters by adverbs $\texttt{adverb}_1$, $\texttt{adverb}_2$, ..., $\texttt{adverb}_n$, then we can write δK_i as

$$\delta K_i = \sum_{j=1}^{m} p_j E_j^i \tag{8.34}$$

where $E_j^i \in \mathbb{R}^{(n+n_1)\times(n+n_1)}$ is a square matrix which has only 1s and 0s as its entries. Let E_j be any of $E_j^i, i = 1, 2, ..., \nu$, then the stable range for designing different adverbs can be guaranteed by the following theorem.

Theorem 8.4.2. *If the following condition is satisfied:*

$$\sum_{j=1}^{m} p_j^2 \leq \frac{\lambda_{\min}^2(Q)}{\sum_{j=1}^{m} \|E_j^\top P + PE_j\|_2^2} \tag{8.35}$$

then the adverb modified controlling verbs are within stable regions. ⊠

Proof. From the condition

$$\sum_{j=1}^{m} p_j^2 \leq \frac{\lambda_{\min}^2(Q)}{\sum_{j=1}^{m} \|E_j^\top P + PE_j\|_2^2} \tag{8.36}$$

we have

$$\sum_{j=1}^{m} p_j^2 \sum_{j=1}^{m} (\|E_j^\top P + PE_j\|_2)^2 \leq \lambda_{\min}^2(Q). \tag{8.37}$$

Since

$$\sum_{j=1}^{m} p_j^2 \sum_{j=1}^{m} (\|E_j^\top P + PE_j\|_2)^2 \geq \sum_{j=1}^{m} (|p_j| \|E_j^\top P + PE_j\|_2)^2 \tag{8.38}$$

we have

$$\sum_{j=1}^{m} (|p_j| \|E_j^\top P + PE_j\|_2) \leq \lambda_{\min}(Q). \tag{8.39}$$

Since

$$\left\| \sum_{j=1}^{m} p_j (E_j^\top P + PE_j) \right\|_2 \leq \sum_{j=1}^{m} (|p_j| \|E_j^\top P + PE_j\|_2) \tag{8.40}$$

we have

$$\left\| \sum_{j=1}^{m} p_j E_j^\top P + P \sum_{j=1}^{m} p_j E_j \right\|_2 \leq \lambda_{\min}(Q) \tag{8.41}$$

followed Theorem 8.4.1 and (8.24) we know that the linear verb control system is asymptotically stable. ∎

8.5 Impulsive Verb Control Based on Basin of Stability

In this kind of impulsive control problem, we assume that the basin of stability of the plant is expressed as a kind of verb knowledge of human experts. Under normal operating conditions, the plant stays within the basin of stability. However, in many applications some system faults can result in abrupt changes of state variables such that the entire plant is kicked outside the basin of stability pseudoimpulsively. In many applications such as the control of transient of power system, the controller needs to captures the plant back into the basin of stability impulsively. In this kind of applications, there is virtually no time for a controller to do computation, instead the knowledge of the basin of stability should be stored as a kind of look-up table and the control signals are generated by simple control rules. This is a kind of impulsive verb control problem whose verb control rule is given by:

IF the plant stays within the basin of stability of target, THEN the controller does nothing; IF the plant leaves the basin of stability of target, THEN the controller captures it back impulsively.

Example 8.5.1. In this example let us design impulsive verb controller to control Van-der-Pol oscillator. This example had been reported as a conventional impulsive control problem in [4]. A Van-der-Pol oscillator with impulse effects induced by system faults is given by

$$\left.\begin{array}{l}\dot{x}(t) = y(t)\\ \dot{y}(t) = -\omega x(t) - \alpha y(t) + \beta y^3(t)\end{array}\right\},\ t \neq \tau_k^F,\ \alpha > 0,\ \beta > 0, \tag{8.42}$$

$$\left.\begin{array}{l}\Delta x = n_x(\tau_k^F)\\ \Delta y = n_y(\tau_k^F)\end{array}\right\},\ t = \tau_k^F,\ k \in \mathbb{N} \tag{8.43}$$

where $\{\tau_k^F\}$ are the moments of the impulse effects induced by system faults, $n_x(\tau_k^F)$ and $n_y(\tau_k^F)$ are the impulsive changes of two state variables caused by the system faults. The stability basin of the system (8.42) is given by

$$\Omega \triangleq \left\{(x, y) \mid x^2 + y^2 \leq \frac{\alpha}{\beta}\right\}. \tag{8.44}$$

Whenever the state variables leave Ω because of the impulsive system faults, the plant is considered unsafe and should be captured back into Ω. There are many different kinds of impulsive control strategies to perform this kind of control, we only present one example as in the following impulsive verb control rules:

IF the plant (8.42) leaves Ω, THEN the controller generates control impulses; IF the controller generates control impulses, THEN control impulses capture the plant (8.42) back into Ω.

The outer system of observing verb leave is given by

$$\mathcal{F}_{\mathsf{leave}}(t) \triangleq \mathcal{F}_{\mathsf{leave}}(x(t), y(t)) = x^2(t) + y^2(t). \tag{8.45}$$

Let $\{\tau_i^C\}$ be the moments of control impulses, then the outer system of generate is given by

$$\begin{array}{l}\mathcal{F}_{\mathsf{generate}} \triangleq \mathcal{F}_{\mathsf{generate}}(\mathcal{F}_{\mathsf{leave}}(t)) :\\ \begin{cases}\tau_i^C = t, & \text{if } \mathcal{F}_{\mathsf{leave}}(t) \geq d\alpha/\beta,\\ \text{no control impulse}, & \text{if } \mathcal{F}_{\mathsf{leave}}(t) < d\alpha/\beta\end{cases}\end{array} \tag{8.46}$$

where $d \in (0, 1)$ is a scaling factor that is used to provide a small time advance for the controller to act before the entire system blows up. The outer system of the controlling verb capture is given by

$$\mathcal{F}_{\mathsf{capture}} \triangleq \mathcal{F}_{\mathsf{capture}}(\tau_i^C) : \begin{cases} \Delta x(\tau_i^C) = -x(\tau_i^C), & \text{if } |x(\tau_i^C)| \geq d\sqrt{\alpha/\beta}, \\ \Delta x(\tau_i^C) = 0, & \text{otherwise}, \\ \Delta y(\tau_i^C) = -y(\tau_i^C), & \text{if } |y(\tau_i^C)| \geq d\sqrt{\alpha/\beta} \\ \Delta y(\tau_i^C) = 0, & \text{otherwise}. \end{cases} \tag{8.47}$$

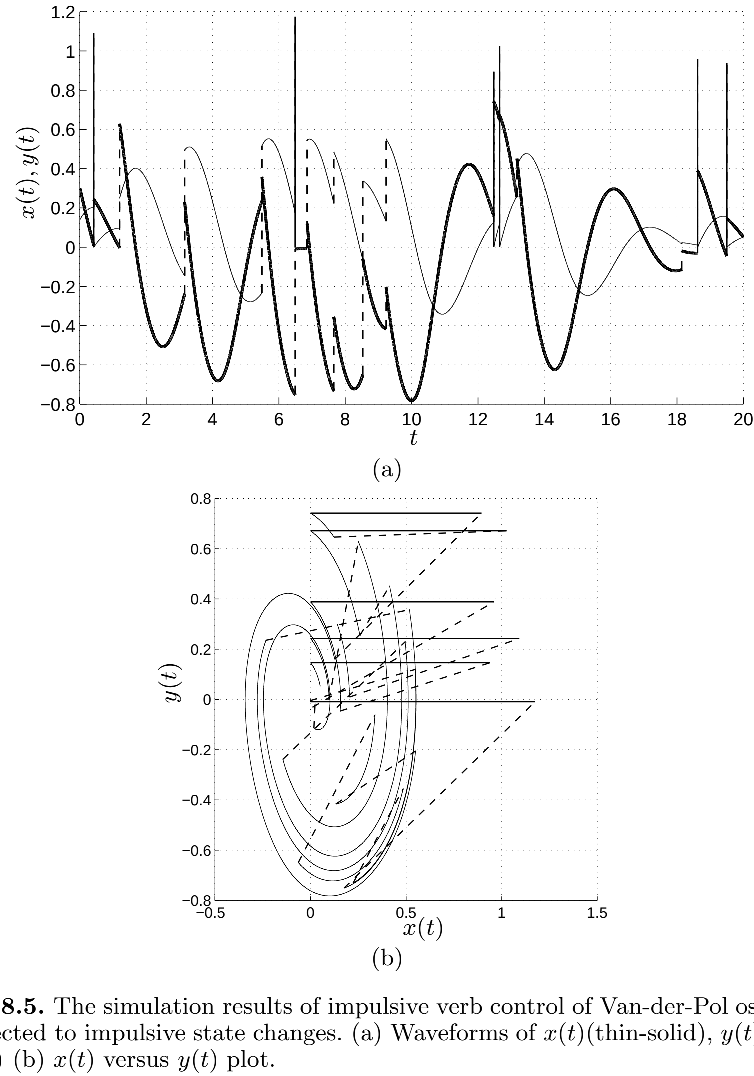


Fig. 8.5. The simulation results of impulsive verb control of Van-der-Pol oscillator subjected to impulsive state changes. (a) Waveforms of $x(t)$(thin-solid), $y(t)$(thick-solid) (b) $x(t)$ versus $y(t)$ plot.

Remark 8.5.1. Although we use crisp representation for the outer systems (8.45), it can also be fuzzy because the computational verb leave can also have a fuzzy collapse[43]. For simplicity, we only use the simplest verb impulse control law (8.47). Some other verb impulse control laws are possible when we introduce different optimal criteria such as minimum power or minimum shock. ♦

We then present some simulation results in Fig. 8.5. The parameters are given by $\alpha = 1$, $\beta = 1$, $\omega = 3$ and $d = 0.8$. Observe that the frequent system faults cause many impulsive state changes, which are plotted as dashed vertical straight lines, to both $x(t)$ and $y(t)$. The verb impulsive control signals are plotted as solid vertical straight lines that may overlap with the dash lines. Observe from Fig. 8.5(b) that the verb impulsive control is quite efficient to keep the plant within the basin of stability. ★

9. Impulsive Control of Periodic Motions

In this chapter we will design impulsive control strategies that can perform the following two kinds of tasks:

1. Stabilize impulsive systems to periodic solutions by using periodic control signals. The applications of this kind of controller to nanoelectronics will be presented in Chapter 11.
2. Using impulsive control signals to stabilize periodic motions.

9.1 Linear Periodic Impulsive Control

Let us first study the cases when plants and impulsive control laws are linear.

9.1.1 Autonomous Cases

Let us study the following linear T-periodic impulsive differential equation:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x}, & t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x}, & t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \qquad (9.1)}$$

where $A \in \mathcal{PC}[\mathbb{R}, \mathbb{C}^{n\times n}]$, $A(t+T) = A(t)$ for $t \in \mathbb{R}$, $B_k \in \mathbb{C}^{n\times n}$, $\det(I+B_k) \neq 0$, $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$. We assume that there is a $\varrho \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ we have

$$B_{k+\varrho} = B_k, \quad \tau_{k+\varrho} = \tau_k. \tag{9.2}$$

Theorem 9.1.1. *The fundamental matrix, $X(t)$, of system (9.1) can be represented in the following form*

$$X(t) = \Theta(t)e^{\Lambda t}, \quad t \in \mathbb{R} \tag{9.3}$$

where $\Theta \in \mathcal{PC}^1[\mathbb{R}, \mathbb{C}^{n\times n}]$ is non-singular and T-periodic, $\Lambda \in \mathbb{C}^{n\times n}$ is a constant matrix. ⊠

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 237−287, 2001.

Proof. Let $Y(t) = X(t+T)$, then $Y(t)$ is a fundamental matrix, we have

$$\begin{aligned}\frac{Y(t)}{dt} &= \frac{X(t+T)}{dt} = A(t+T)X(t+T) = A(t)Y(t), \quad t \neq \tau_k, \\ \Delta Y(\tau_k) &= \Delta X(\tau_k + T) = \Delta X(\tau_{k+\varrho}) \\ &= B_{k+\varrho}X(\tau_{k+\varrho}) = B_k X(\tau_k + T) = B_k Y(\tau_k).\end{aligned} \tag{9.4}$$

From the second conclusion of Theorem 1.6.2 we know that there is a unique non-single matrix $M \in \mathbb{C}^{n\times n}$ (called the *monodromy matrix* of system (9.1)) such that

$$X(t+T) = X(t)M. \tag{9.5}$$

Let us choose

$$\begin{aligned}\Lambda &= \frac{1}{T}\ln M, \\ \Theta(t) &= X(t)e^{-\Lambda t}\end{aligned} \tag{9.6}$$

then it is easy to see that (9.3) holds, $\Theta(t)$ is non-singular, $\Theta \in \mathcal{PC}^1[\mathbb{R}, \mathbb{C}^{n\times n}]$ and $Me^{-\Lambda T} = I$. We then have

$$\Theta(t+T) = X(t+T)e^{-\Lambda(t+T)} = X(t)Me^{-\Lambda T}e^{-\Lambda t} = \Theta(t), \tag{9.7}$$

which proves that $\Theta(t)$ is T-periodic. ■

Remark 9.1.1. If system (9.1) is real and let us define

$$\Lambda_2 \triangleq \frac{1}{2T}\ln M^2, \quad \Theta_2(t) \triangleq X(t)e^{-\Lambda_2 t}, \tag{9.8}$$

then we have

$$X(t) = \Theta_2(t)e^{\Lambda_2 t}, \tag{9.9}$$

where Λ_2 and $\Theta_2(t)$ are real and $\Theta_2(t)$ is $2T$-periodic. ♦

Theorem 9.1.2. *All monodromy matrices of system (9.1) are similar and therefore have the same eigenvalues.* ⊠

Proof. Let M_1 and M_2 are two monodromy matrices of system (9.1) with respect to fundamental matrices $X_1(t)$ and $X_2(t)$, respectively. We then have

$$X_1(t+T) = X_1(t)M_1, \quad X_2(t+T) = X_2(t)M_2. \tag{9.10}$$

Then from the second conclusion of Theorem 1.6.2 we have

$$X_2(t) = X_1(t)S, \quad X_2(t+T) = X_1(t+T)S, \quad \det S \neq 0 \tag{9.11}$$

from which we have

$$
\begin{aligned}
X_1(t+T)S &= X_2(t+T) \\
&= X_2(t)M_2 \qquad \Leftarrow (9.10) \\
&= X_1(t)SM_2 \qquad \Leftarrow (9.11).
\end{aligned} \tag{9.12}
$$

Then from (9.10) and (9.12) we have

$$
X_1(t)M_1S = X_1(t)SM_2, \tag{9.13}
$$

from which we have

$$
M_1S = SM_2, \tag{9.14}
$$

which leads to

$$
M_1 = SM_2S^{-1} \tag{9.15}
$$

because S is non-singular. This proves that all monodromy matrices of system (9.1) are similar. Since similar matrices have same eigenvalues, we complete the proof. ■

The eigenvalues $\mu_1, \cdots, \mu_n$ of the monodromy matrix are called *multipliers* of system (9.1) and the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the matrix Λ are called *characteristic exponents* and it follows from (9.6) that

$$
\lambda_i = \frac{1}{T} \ln \mu_i, \quad i = 1, \cdots, n. \tag{9.16}
$$

For any monodromy matrix, M, of system (9.1) we have

$$
\det M = \prod_{i=1}^{n} \mu_i = \prod_{i=1}^{\varrho} \det(I + B_i) \exp\left(\int_0^T \mathrm{Tr} A(s)ds\right). \tag{9.17}
$$

As a direct consequence to Theorem 9.1.1 we have the following corollary.

Corollary 9.1.1.

1. *$\mu \in \mathbb{C}$ is a multiplier of system (9.1) if and only if there is a non-trivial solution, $\xi(t)$, of system (9.1) such that*

$$
\xi(t+T) = \mu\xi(t), \quad t \in \mathbb{R};
$$

2. *System (9.1) has a nontrivial pT-periodic solution if and only if the pth power of some of its multipliers are equal to 1.*

⊠

Theorem 9.1.3. *System (9.1) can be reduced to*

$$
\dot{\boldsymbol{y}} = \Lambda \boldsymbol{y} \tag{9.18}
$$

by using transformation

$$
\boldsymbol{x} = \Theta(t)\boldsymbol{y}. \tag{9.19}
$$

⊠

Proof. It follows from Theorem 9.1.1 that

$$X(t) = \Theta(t)e^{\Lambda t}. \tag{9.20}$$

By using transformation (9.19) we have, for $t \neq \tau_k$

$$\dot{\Theta}(t)\boldsymbol{y} + \Theta(t)\dot{\boldsymbol{y}} = A(t)\boldsymbol{x} = A(t)\Theta(t)\boldsymbol{y} \tag{9.21}$$

from which and (9.20) we have

$$\begin{aligned}
&\dot{X}(t)e^{-\Lambda t}\boldsymbol{y} - X(t)\Lambda e^{-\Lambda t}\boldsymbol{y} + X(t)e^{-\Lambda t}\dot{\boldsymbol{y}} = A(t)X(t)e^{-\Lambda t}\boldsymbol{y}\\
&A(t)X(t)e^{-\Lambda t}\boldsymbol{y} - X(t)\Lambda e^{-\Lambda t}\boldsymbol{y} + X(t)e^{-\Lambda t}\dot{\boldsymbol{y}} = A(t)X(t)e^{-\Lambda t}\boldsymbol{y},\\
&X(t)e^{-\Lambda t}\dot{\boldsymbol{y}} = X(t)\Lambda e^{-\Lambda t}\boldsymbol{y},\\
&e^{-\Lambda t}\dot{\boldsymbol{y}} = \Lambda e^{-\Lambda t}\boldsymbol{y},\\
&\dot{\boldsymbol{y}} = e^{\Lambda t}\Lambda e^{-\Lambda t}\boldsymbol{y} = \Lambda \boldsymbol{y}
\end{aligned} \tag{9.22}$$

from which we immediate have (9.18).

For $t = \tau_k$ we have

$$\Theta(\tau_k^+)\boldsymbol{y}(\tau_k^+) - \Theta(\tau_k)\boldsymbol{y}(\tau_k) = B_k\Theta(\tau_k)\boldsymbol{y}(\tau_k), \tag{9.23}$$

from which and (9.20) we have

$$X(\tau_k^+)e^{-\Lambda\tau_k}\boldsymbol{y}(\tau_k^+) - X(\tau_k)e^{-\Lambda\tau_k}\boldsymbol{y}(\tau_k) = B_kX(\tau_k)e^{-\Lambda\tau_k}\boldsymbol{y}(\tau_k). \tag{9.24}$$

In view of $X(\tau_k^+) = (I + B_k)X(\tau_k)$ we have

$$X(\tau_k^+)e^{-\Lambda\tau_k}\boldsymbol{y}(\tau_k^+) = X(\tau_k^+)e^{-\Lambda\tau_k}\boldsymbol{y}(\tau_k) \tag{9.25}$$

which leads to

$$\boldsymbol{y}(\tau_k^+) = \boldsymbol{y}(\tau_k). \tag{9.26}$$

Therefore, the impulse effects disappear by using transformation (9.19). This completes the proof. ■

We then have the main theorem for the stability of system (9.1) as follows.

Theorem 9.1.4. *System (9.1) is*

1. *stable if and only if all of its multipliers μ_i, $i = 1, \cdots, n$, satisfy*

$$|\mu_i| \leq 1$$

   and those μ_i with $|\mu_i| = 1$ are simple;
2. *asymptotically stable if and only if all of its multipliers μ_i, $i = 1, \cdots, n$, satisfy*

$$|\mu_i| < 1;$$

3. *unstable if*

$$|\mu_i| > 1$$

for some $i = 1, \cdots, n$.

⊠

Proof. We have the following relation:

$$\frac{1}{T}\ln|\mu_i| = \mathrm{Re}\lambda_i, \quad i = 1, \cdots, n. \tag{9.27}$$

Then it follows from Theorem 9.1.3 we immediately have the conclusions. ■

Remark 9.1.2. From Theorem 9.1.4 we know that it is important to find the multipliers in order to determine the stability properties of system (9.1). In order to do so, let us first fix a $t_0 \in \mathbb{R}$ and choose an arbitrary fundamental matrix $X(t)$ of system (9.1) and then find the eigenvalue of the following matrix

$$M = \Psi(t_0 + T, t_0) = X(t_0 + T)X^{-1}(t_0). \tag{9.28}$$

There are two useful special cases listed as follows.

1. if $X(0) = I$, then we can choose $M = X(T)$;
2. if $X(0^+) = I$, then we can choose $M = X(T^+)$.

♦

Example 9.1.1. Consider the following linear periodic impulsive control system

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}}_{A}, \quad t \neq \tau_0 + kT, \\
\Delta\begin{pmatrix} x \\ y \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 0 \\ a/\omega & b \end{pmatrix}}_{B}\begin{pmatrix} x \\ y \end{pmatrix}, \quad t = \tau_0 + kT, \quad k \in \mathbb{Z}, \quad T > 0.
\end{aligned} \tag{9.29}$$

It follows from Remark 9.1.2 that the monodromy matrix of system (9.29) is given by

$$\begin{aligned}
M &= \Psi(\tau_0 + T^+, \tau_0^+) = (I + B)e^{AT} \\
&= \begin{pmatrix} 1 & 0 \\ a/\omega & b+1 \end{pmatrix}\begin{pmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\omega T) & \sin(\omega T) \\ \frac{a}{\omega}\cos(\omega T) - (b+1)\sin(\omega T) & \frac{a}{\omega}\sin(\omega T) + (b+1)\cos(\omega T) \end{pmatrix}
\end{aligned} \tag{9.30}$$

whose eigenvalues are the multipliers μ_p, $p = 1, 2$ that are given by the roots of

$$\mu^2 - \left(\frac{a}{\omega}\sin(\omega T) + (b+2)\cos(\omega T)\right)\mu + b + 1 = 0. \tag{9.31}$$

Then we consider the following cases.

1. T-periodic solutions.
   If at least one root of (9.31) is equal to 1, then system (9.29) had non-trivial T-periodic solution. In this case, in view of (9.31), we have the following condition:

$$\frac{a}{\omega}\sin(\omega T) + (b+2)\cos(\omega T) = b + 2. \tag{9.32}$$

   If we need to design impulsive control laws such that the T-periodic solution is stable, the second multiplier should be less than one, in this case, we have the following condition

$$|b+1| < 1. \tag{9.33}$$

   Then from (9.32) and (9.33) we can choose the parameters for the impulsive controller to achieve an asymptotically stable T-periodic solution. We will show that the controlled system (9.29) may have infinite many stable T-periodic solutions. Let $\boldsymbol{x}(t) = (x(t), y(t))^\top$ be a T-periodic solution of system (9.29) with initial condition $\boldsymbol{x}_0 \triangleq (x(\tau_0^+), y(\tau_0^+))^\top = (x_0, y_0)^\top$, then we have an additional condition to define this T-periodic solution; namely, $M\boldsymbol{x}_0 = \boldsymbol{x}_0$, from which we have

$$\begin{aligned} &[\cos(\omega T) - 1]x_0 + \sin(\omega T)y_0 = 0, \\ &\left(\frac{a}{\omega}\cos(\omega T) - (b+1)\sin(\omega T)\right)x_0 \\ &+\left(\frac{a}{\omega}\sin(\omega T) + (b+1)\cos(\omega T) - 1\right)y_0 = 0. \end{aligned} \tag{9.34}$$

   We have the following three kinds of situations:
   a) $\omega T = 2p\pi$, $p \in \mathbb{N}$.
      Condition (9.32) holds and (9.34) becomes the following stable initial condition line[1]:

$$\frac{a}{\omega}x_0 + by_0 = 0. \tag{9.35}$$

      For $t \in (\tau_k, \tau_{k+1})$ we have

$$\begin{aligned} x(t) &= x_0\cos\omega(t-\tau_k) + y_0\sin\omega(t-\tau_k), \\ y(t) &= -x_0\sin\omega(t-\tau_k) + y_0\cos\omega(t-\tau_k), \end{aligned} \tag{9.36}$$

[1] Any trajectory starting from an initial point located on the stable initial line will generate a stable periodic solution.

therefore $x(\tau_{k+1}) = x_0$ and $y(\tau_{k+1}) = y_0$. This means that at $t = \tau_{k+1}$ the impulsive control is zero.
In the simulation shown in Fig. 9.1 we set $\omega = \pi$, $T = 2$, $a = 1$, $b = -1.5$. The initial conditions for the solid and dashed curves are given by $(x_0, y_0) = (2, -3)$ and $(x_0, y_0) = (-2, -2)$, respectively. Observe that the trajectories approach two limit cycles. The dotted straight line is the stable initial condition line.

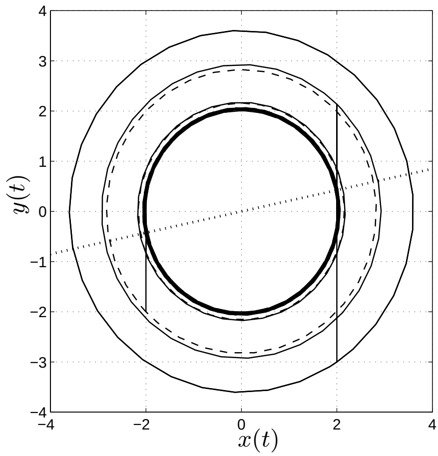


Fig. 9.1. The simulation results of controlling periodic impulsive systems: Case 1.

b) $\omega T \neq 2p\pi, p \in \mathbb{N}$, $a = 0$ and $b = -2$.
Condition (9.32) holds for all $T > 0$ and $\omega T \neq 2p\pi$. In this case we have a multiple multiplier at $\mu_{1,2} = 1$, condition (9.33) can not be satisfied. Therefore the periodic trajectory is not asymptotically stable. From (9.34) we have

$$\begin{aligned}
&[\cos(\omega T) - 1]x_0 + \sin(\omega T)y_0 = 0, \\
&\sin(\omega T)x_0 - [\cos(\omega T) + 1]y_0 = 0,
\end{aligned} \tag{9.37}$$

from which we have

$$\begin{aligned}
&[\cos(\omega T) - 1]\frac{\cos(\omega T) + 1}{\sin(\omega T)}y_0 + \sin(\omega T)y_0 = 0, \\
&x_0 = \frac{\cos(\omega T) + 1}{\sin(\omega T)}y_0,
\end{aligned} \tag{9.38}$$

which leads to the stable initial condition line:

$$x_0 = \frac{\cos(\omega T) + 1}{\sin(\omega T)}y_0, \tag{9.39}$$

which is equivalent to

$$-x_0 + \cot(\omega T/2)y_0 = 0. \tag{9.40}$$

In the simulation shown in Fig. 9.2 we set $\omega = 3$, $T = 2$, $a = 0$, $b = -2$. The initial conditions for the solid and dashed curves are given by $(x_0, y_0) = (2, -0.2818)$(on the stable initial condition line) and $(x_0, y_0) = (-2, -2)$(not on the stable initial condition line), respectively. Observe that only the solid trajectory forms a T-periodic. Since the initial condition of the dashed trajectory is not on the stable initial condition line, the observed solution is not T-periodic. This can be verified by the two positions of impulses shown as two vertical straight dashed lines. Also, the stable initial condition line is shown as a dotted line.

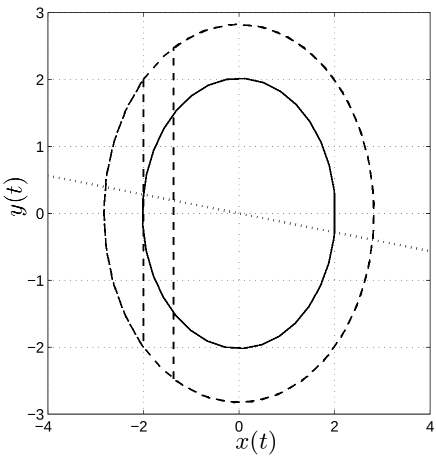


Fig. 9.2. The simulation results of controlling periodic impulsive systems: Case 2.

c) $\omega T \neq 2p\pi, p \in \mathbb{N}$, $a \neq 0$ and from (9.34) we know that the stable initial condition line is given by

$$y_0 = -\frac{\cos(\omega T) - 1}{\sin(\omega T)} x_0 \tag{9.41}$$

with parameter satisfying

$$\begin{aligned}&\left(\frac{a}{\omega}\cos(\omega T) - (b+1)\sin(\omega T)\right)\sin(\omega T)\\ &= \left(\frac{a}{\omega}\sin(\omega T) + (b+1)\cos(\omega T) - 1\right)[\cos(\omega T) - 1]\end{aligned}$$

and (9.33).

In the simulation shown in Fig. 9.3 we set $\omega = 3$, $T = 2$, $a = -0.2$, $b = -1.5323$. The initial conditions for the solid and dashed curves are given by $(x_0, y_0) = (2, -3)$ and $(x_0, y_0) = (-2, -2)$, respectively. Observe that the trajectories approach two limit cycles. The dotted straight line is the stable initial condition line. Observe that the control impulses make $(x(\tau_k^+), y(\tau_k^+))$ approach a point on the stable initial condition line.

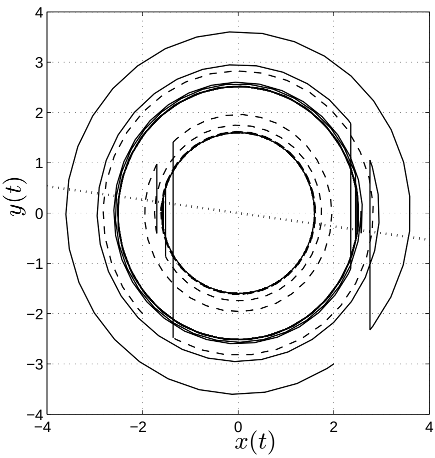


Fig. 9.3. The simulation results of controlling periodic impulsive systems: Case 3.

2. $2T$-periodic solutions.
   To study $2T$-periodic solutions, we need to consider the eigenvalues ν_p, $p = 1, 2$, of matrix M^2. It is easy to see that $\nu_p = \mu_p^2$, $p = 1, 2$, then from (9.31) we have

$$\begin{aligned} \nu_1 + \nu_2 &= \mu_1^2 + \mu_2^2 = (\mu_1 + \mu_2)^2 - 2\mu_1\mu_2 \\ &= \left(\frac{a}{\omega}\sin(\omega T) + (b+2)\cos(\omega T)\right)^2 - 2(b+1), \\ \nu_1\nu_2 &= \mu_1^2\mu_2^2 = (b+1)^2. \end{aligned} \tag{9.42}$$

   Therefore, ν_p, $p = 1, 2$ are roots of the following equation:

$$\begin{aligned} &\nu^2 - \left[\left(\frac{a}{\omega}\sin(\omega T) + (b+2)\cos(\omega T)\right)^2 - 2(b+1)\right]\nu \\ &+(b+1)^2 = 0. \end{aligned} \tag{9.43}$$

   System (9.29) has non-trivial $2T$-periodic solutions if and only of $\nu = 1$ is a solution of (9.43), thus we have

$$\left(\frac{a}{\omega}\sin(\omega T) + (b+2)\cos(\omega T)\right)^2 = (b+2)^2. \tag{9.44}$$

Then the stable initial condition for $2T$-periodic solutions are given by $M^2\boldsymbol{x}_0 = \boldsymbol{x}_0$; namely,

$$\begin{aligned}
&\sin(\omega T)\left(\frac{a}{\omega}x_0 - (b+2)y_0\right) = 0,\\
&(b+2)\left(\frac{a}{\omega}\cos(\omega T) - (b+1)\sin(\omega T)\right)x_0\\
&+\left((b+1)\frac{a}{\omega}\sin(\omega T) + b(b+2)\cos(\omega T)\right)y_0 = 0.
\end{aligned} \tag{9.45}$$

The rest of analysis is similar to that of T-periodic cases.

★

9.1.2 Nonautonomous Cases

In this section let us consider the following impulsive control system with impulses at fixed time:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t),\ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x} + \boldsymbol{c}_k, \quad t = \tau_k, \quad k \in \mathbb{Z} \end{cases}} \tag{9.46}$$

where $A \in \mathcal{PC}[\mathbb{R}, \mathbb{C}^{n\times n}]$, $A(t+T) = A(t)$ for $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathcal{PC}[\mathbb{R}, \mathbb{C}^n]$, $\boldsymbol{u}(t+T) = \boldsymbol{u}(t)$, $B_k \in \mathbb{C}^{n\times n}$, $\det(I + B_k) \neq 0$, $\boldsymbol{c}_k \in \mathbb{C}^n$, $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$. We assume that there is a $\varrho \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ we have

$$B_{k+\varrho} = B_k, \quad \boldsymbol{c}_{k+\varrho} = \boldsymbol{c}_k, \quad \tau_{k+\varrho} = \tau_k. \tag{9.47}$$

Let $X(t) = \Psi(t, 0)$ be the normalized fundamental matrix of system (9.1) at $t = 0$, then the solution, $\boldsymbol{x}(t) = \boldsymbol{x}(t, 0, \boldsymbol{x}_0)$, of system (9.46) is given by

$$\begin{aligned}
\boldsymbol{x}(t) = X(t)\boldsymbol{x}_0 &+ \int_0^t X(t)X^{-1}(s)\boldsymbol{u}(s)ds\\
&+ \sum_{0\leq\tau_k<t} X(t)X^{-1}(\tau_k^+)\boldsymbol{c}_k
\end{aligned} \tag{9.48}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(0)$. If $\boldsymbol{x}(t)$ is T-periodic, then we have $\boldsymbol{x}(T) = \boldsymbol{x}(0)$; namely,

$$\begin{aligned}
[I - X(T)]\boldsymbol{x}(0) = &\int_0^T X(T)X^{-1}(s)\boldsymbol{u}(s)ds\\
&+ \sum_{0\leq\tau_k<T} X(T)X^{-1}(\tau_k^+)\boldsymbol{c}_k.
\end{aligned} \tag{9.49}$$

Case $\det[I - X(t)] \neq 0$.

Definition 9.1.1. *Let us define $\Upsilon(t,s)$ as*

$$\Upsilon(t,s) \triangleq \begin{cases} X(t)[I-X(T)]^{-1}X^{-1}(s), & 0 < s < t \leq T, \\ X(t+T)[I-X(T)]^{-1}X^{-1}(s), & 0 < t \leq s \leq T, \\ \Upsilon(t-kT, s-lT), & kT < t \leq (k+1)T, \\ & lT < s \leq (l+1)T,\ k,l \in \mathbb{Z}, \end{cases} \tag{9.50}$$

which is called the Green's function for the periodic solutions of system (9.46).

⊠

Remark 9.1.3. $\Upsilon(t,s)$ has the following properties

1. $\Upsilon(t,t^-) - \Upsilon(t,t^+) = I$ for $t \neq \tau_k$ and $t \in \mathbb{R}$;
2. $\Upsilon(t+T,s) = \Upsilon(t,s)$ for $t \in \mathbb{R}$ and $s \in \mathbb{R}$;
3. $$\frac{\partial \Upsilon(t,s)}{\partial t} = A(t)\Upsilon(t,s)$$
   for $t \neq \tau_k$ and $s \in \mathbb{R}$;
4. $\Upsilon(\tau_k^+, s) = (I + B_k)\Upsilon(\tau_k, s)$ for $s \neq \tau_k$;
5. $$\Upsilon(\tau_k^+, \tau_k^+) = \lim_{t \to \tau_k^+} \Upsilon(t, \tau_k^+) = (I + B_k)\Upsilon(\tau_k, \tau_k^+) + I.$$

♦

Theorem 9.1.5. *Assume that $I - X(t)$ is non-singular and system (9.1) has no non-trivial T-periodic solution, then system (9.46) has a unique T-periodic solution*

$$\boldsymbol{x}_T(t) = \int_0^T \Upsilon(t,s)\boldsymbol{u}(s)ds + \sum_{0 \leq \tau_k < T} \Upsilon(t, \tau_k^+)\boldsymbol{c}_k, \tag{9.51}$$

which satisfies

$$\|\boldsymbol{x}_T(t)\| \leq L \left(\int_0^T \|\boldsymbol{u}(s)\| ds + \sum_{k=1}^{\varrho} \|\boldsymbol{c}_k\| \right), \tag{9.52}$$

where

$$L = \sup_{t,s \in [0,T]} \|\Upsilon(t,s)\|. \tag{9.53}$$

⊠

Proof. Since $I - X(t)$ is non-singular we know that $\det[I - X(t)] \neq 0$ and $M = X(T)$ is a monodromy matrix of system (9.1), we know that all multipliers of system (9.1) are not equal to 1. Furthermore, this means that system (9.1) does not have non-trivial T-periodic solution. Then, (9.49) has a unique solution

$$\begin{aligned}\boldsymbol{x}(0) = [I - X(T)]^{-1} \Bigg(&\int_0^T X(T)X^{-1}(s)\boldsymbol{u}(s)ds \\ &+ \sum_{0 \leq \tau_k < T} X(T)X^{-1}(\tau_k^+)\boldsymbol{c}_k \Bigg). \end{aligned} \tag{9.54}$$

Therefore system (9.46) has a unique T-periodic solution

$$\begin{aligned}\boldsymbol{x}_T(t) = X(t)[I - X(T)]^{-1} \Bigg(&\int_0^T X(T)X^{-1}(s)\boldsymbol{u}(s)ds \\ &+ \sum_{0 \leq \tau_k < T} X(T)X^{-1}(\tau_k^+)\boldsymbol{c}_k \Bigg) \\ &+ \int_0^t X(t)X^{-1}(s)\boldsymbol{u}(s)ds + \sum_{0 \leq \tau_k < t} X(t)X^{-1}(\tau_k^+)\boldsymbol{c}_k, \end{aligned} \tag{9.55}$$

which is exactly (9.51) and from which we immediately have the estimate in (9.52). ■

Remark 9.1.4. If all multipliers, $\mu_i, i = 1, \cdots, n$, of system (9.1) satisfy $|\mu_i| < 1$, then $\boldsymbol{x}_T(t)$ is exponentially stable. ♦

Case $\det[\boldsymbol{I} - \boldsymbol{X(t)}] = \mathbf{0}$. In this case, system (9.1) has non-trivial T-periodic solutions. Let us construct the following adjoint equation of (9.1)

$$\begin{cases} \dot{\boldsymbol{y}} = -A^*(t)\boldsymbol{y}, & t = \tau_k, \\ \Delta\boldsymbol{y} = -(I + B_k^*)^{-1}B_k^*\boldsymbol{y}, & t = \tau_k, \quad k \in \mathbb{Z}. \end{cases} \tag{9.56}$$

The following lemma will be used in the proof of Theorem 9.1.6. Otherwise, it also provides some conclusions on the relations between the solutions and fundamental matrices of the mutually adjoint equations (9.1) and (9.56).

Lemma 9.1.1. *Assume that $A \in \mathcal{PC}[\mathbb{R}, \mathbb{C}^{n\times n}]$, $B_k \in \mathbb{C}^{n\times n}$ and $\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$, then we have the following conclusions:*

1. *Let $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ be any solutions of mutually adjoint equations (9.1) and (9.56), respectively, then for $t \in \mathbb{R}$ we have*

$$\boldsymbol{x}^*(t)\boldsymbol{y}(t) = \boldsymbol{x}^*(0)\boldsymbol{y}(0) = \varsigma \tag{9.57}$$

where ς is a constant.

2. *Let $X(t)$ and $Y(t)$ be fundamental matrices of mutually adjoint equations (9.1) and (9.56), respectively, then for $t \in \mathbb{R}$ we have*

$$Y^*(t)X(t) = C, \tag{9.58}$$

where $C \in \mathbb{C}^{n\times n}$ is a constant matrix.

3. *If $X(t)$ is a fundamental matrix of (9.1), $C \in \mathbb{C}^{n\times n}$ is a nonsingular constant matrix and (9.58) holds, then $Y(t)$ is a fundamental matrix of (9.56).*

$\boxtimes$

Proof. Conclusion 1: For $t \in (\tau_k, \tau_{k+1}]$ we have

$$\begin{aligned}
\frac{d\boldsymbol{x}^*(t)\boldsymbol{y}(t)}{dt} &= \dot{\boldsymbol{x}}^*(t)\boldsymbol{y}(t) + \boldsymbol{x}^*(t)\dot{\boldsymbol{y}}(t) \\
&= [A(t)\boldsymbol{x}(t)]^*\boldsymbol{y}(t) - \boldsymbol{x}^*(t)A^*(t)\boldsymbol{y}(t) \\
&= [A(t)\boldsymbol{x}(t)]^*\boldsymbol{y}(t) - [A(t)\boldsymbol{x}(t)]^*\boldsymbol{y}(t) = 0.
\end{aligned} \tag{9.59}$$

Therefore $\boldsymbol{x}^*(t)\boldsymbol{y}(t) = \varsigma_k$ for each $k \in \mathbb{Z}$ and $t \in (\tau_k, \tau_{k+1}]$. For $t = \tau_k^+$, $k \in \mathbb{Z}$ we have

$$\begin{aligned}
\boldsymbol{x}^*(\tau_k^+)\boldsymbol{y}(\tau_k^+) &= [(I + B_k)\boldsymbol{x}(\tau_k)]^*[I - (I + B_k^*)^{-1}B_k^*]\boldsymbol{y}(\tau_k) \\
&= [(I + B_k)\boldsymbol{x}(\tau_k)]^*(I + B_k^*)^{-1}\boldsymbol{y}(\tau_k) \\
&= \boldsymbol{x}^*(\tau_k)\boldsymbol{y}(\tau_k), \quad k \in \mathbb{Z},
\end{aligned} \tag{9.60}$$

then we know that $\varsigma_k = \varsigma$ for all $k \in \mathbb{Z}$.
Conclusion 2: Can be proved by using the similar procedure as that for conclusion 1.
Conclusion 3: For $t \neq \tau_k, k \in \mathbb{Z}$, it follows from (9.58) that

$$\frac{dY(t)}{dt} = -A^*(t)Y(t). \tag{9.61}$$

For $t = \tau_k, k \in \mathbb{Z}$ we have

$$\begin{aligned}
Y(\tau_k^+) &= [X^*(\tau_k^+)]^{-1}C^* \\
&= (I + B_k^*)^{-1}[X^*(\tau_k)]^{-1}C^* \\
&= (I + B_k^*)^{-1}Y(\tau_k),
\end{aligned} \tag{9.62}$$

from which we have

$$\Delta Y(\tau_k) = [(I + B_k^*)^{-1} - I]Y(\tau_k) = -(I + B_k^*)^{-1}B_k^*Y(\tau_k). \tag{9.63}$$

This proves that $Y(t), t \in \mathbb{R}$ is a matrix solution of system (9.56). Since $Y(t), t \in \mathbb{R}$ is nonsingular, it is a fundamental matrix of system (9.56). $\blacksquare$

The following lemma will be used in the proofs of Theorems 9.1.6 and 9.1.7. It can be easily proved based on standard matrix theory.

Lemma 9.1.2. *Assume that*

$$A\boldsymbol{x} = \boldsymbol{a}, \tag{9.64}$$

$$A\boldsymbol{y} = 0, \tag{9.65}$$

$$A^*\boldsymbol{z} = 0, \tag{9.66}$$

where $A \in \mathbb{C}^{n\times n}$, $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{C}^n$, *then*

1. *The mutually adjoint equations (9.1) and (9.56) have the same number of linearly independent solutions and*

$$m = n - \mathrm{rank}\ A = n - \mathrm{rank}\ A^*.$$

2. *If* $\boldsymbol{z}_1, \cdots, \boldsymbol{z}_m$ *are* m *linearly independent solutions of (9.66), then (9.64) has a solution if and only if* $\boldsymbol{z}_i^*\boldsymbol{a} = 0$, $i = 1, \cdots, m$.
3. *If* $\boldsymbol{z}_i^*\boldsymbol{a} = 0$, $i = 1, \cdots, m$, *then there is a unique solution* $\boldsymbol{x}_1$ *of (9.64) such that* $\boldsymbol{y}_i^*\boldsymbol{x}_1 = 0$ *for* $i = 1, \cdots, m$, *where* $\boldsymbol{y}_i$, $i = 1, \cdots, m$ *are linearly independent solutions of (9.65). Let* $Z \triangleq (\boldsymbol{z}_1, \cdots, \boldsymbol{z}_m) \in \mathbb{C}^{n\times m}$, $Y \triangleq (\boldsymbol{y}_1, \cdots, \boldsymbol{y}_m) \in \mathbb{C}^{n\times m}$, *and* $B \triangleq A - ZY^*$, *then we have*

$$\boldsymbol{x}_1 = B^{-1}\boldsymbol{a}.$$

Furthermore, B *is nonsingular and there is a constant* $L > 0$ *independent of* $\boldsymbol{a}$ *such that* $\|\boldsymbol{x}_1\| \le L\|\boldsymbol{a}\|$.

⊠

Theorem 9.1.6. *Let system (9.1) have* m *linearly independent* T*-periodic solutions* $\boldsymbol{\varphi}_1(t), \cdots, \boldsymbol{\varphi}_m(t)$ *with* $1 \le m \le n$, *and let* $\Psi(t,s)$ *be the Cauchy matrix of system (9.1) and*

$$\begin{aligned} &M = \Psi(t,0), \qquad \Gamma_\psi(t) = (\boldsymbol{\psi}_1(t), \cdots, \boldsymbol{\psi}_m(t)), \\ &\Gamma_\varphi(t) = (\boldsymbol{\varphi}_1(t), \cdots, \boldsymbol{\varphi}_m(t)), \end{aligned} \tag{9.67}$$

then

1. *System (9.56) has* m *linearly independent* T*-periodic solutions* $\boldsymbol{\psi}_1(t), \cdots, \boldsymbol{\psi}_m(t)$;
2. *System (9.46) has a* T*-periodic solution if and only if*

$$\int_0^T \Gamma_\psi^*(t)\boldsymbol{g}(t)dt + \sum_{0\le\tau_k<T} \Gamma_\psi^*(\tau_k^+)\boldsymbol{c}_k = 0. \tag{9.68}$$

Furthermore, let $\boldsymbol{x}_a(t)$ *be a particular* T*-periodic solution of system (9.46), then each* T*-periodic solution of system (9.46) has the form*

$$\boldsymbol{x}(t) = \boldsymbol{x}_a(t) + \sum_{i=1}^m \varsigma_i\boldsymbol{\varphi_i(t)}, \tag{9.69}$$

where $\varsigma_i, i = 1, \cdots, m$ *are constants;*

3. *If (9.68) holds, then system (9.46) has a unique T-period solution $\boldsymbol{x}_T(t)$, which satisfies the condition*

$$\Gamma_\varphi^*(0)\boldsymbol{x}_T(0) = 0, \tag{9.70}$$

and is given by

$$\boldsymbol{x}_T(t) = \int_0^T \tilde{\Upsilon}(t,s)\boldsymbol{u}(s)ds + \sum_{i=1}^{\varrho} \tilde{\Upsilon}(t,\tau_i^+)\boldsymbol{c}_i, \tag{9.71}$$

where

$$\tilde{\Upsilon}(t,s) = \begin{cases} \Psi(t,0)P^{-1}\Psi(T,s) + \Psi(t,s), & 0 < s < t \le T, \\ \Psi(t,0)P^{-1}\Psi(T,s), & 0 < t \le s \le T, \\ \tilde{\Upsilon}(t-kT, s-lT), & kT < t \le (k+1)T, \\ & lT < s \le (l+1)T, \\ & k,l \in \mathbb{Z} \end{cases} \tag{9.72}$$

is the generalized Green's function and

$$P \triangleq (I - M - \Gamma_\psi(0)\Gamma_\varphi^*(0)) \tag{9.73}$$

is nonsingular.
Furthermore, there is a constant $L > 0$ independent of $\boldsymbol{u}(t)$ and $\boldsymbol{c}_k$ such that

$$\sup_{t\in\mathbb{R}} \|\boldsymbol{x}_T(t)\| \le L\left(\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\| + \max_{k=1}^{\varrho} \|\boldsymbol{c}_k\|\right). \tag{9.74}$$

⊠

Proof. Conclusion 1: Since system (9.1) have m linearly independent T-periodic solutions $\boldsymbol{\varphi}_1(t), \cdots, \boldsymbol{\varphi}_m(t)$ with $1 \le m \le n$, we know that

$$(I - M)\boldsymbol{x} = 0 \tag{9.75}$$

has m linearly independent solutions $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m$ such that $\boldsymbol{\varphi}_i(0) = \boldsymbol{x}_i$, $i = 1, \cdots, m$. Then from Lemma 9.1.2 we know that the following adjoint equation of (9.75)

$$(I - M^*)\boldsymbol{y} = 0 \tag{9.76}$$

also has m linearly independent solutions $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_m$ such that there are m linearly independent T-periodic solutions $\boldsymbol{\psi}_1(t), \cdots, \boldsymbol{\psi}_m(t)$ of system (9.56) satisfying $\boldsymbol{\psi}_i(0) = \boldsymbol{y}_i$, $i = 1, \cdots, m$.
Conclusion 2: System (9.46) has a T-periodic solution $\boldsymbol{x}(t)$ if and only if the equation

$$(I - M)\boldsymbol{x}(0) = \int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{k=1}^{\varrho} \Psi(T,\tau_k^+)\boldsymbol{c}_k \triangleq \boldsymbol{a} \tag{9.77}$$

has a solution $\boldsymbol{x}(0)$. From conclusion 3 of Lemma 9.1.2 we know that the above condition is equivalent to

$$\boldsymbol{y}_i^* \boldsymbol{a} = 0 \text{ for } i = 1, \cdots, m, \tag{9.78}$$

or in the following matrix form

$$\Gamma_\psi^*(0) \left(\int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0 \le \tau_k < T} \Psi(T,\tau_k^+)\boldsymbol{c}_k \right) = 0. \tag{9.79}$$

Let $\Phi(t) = (\Gamma_\varphi(t), \tilde{\Gamma}_\varphi(t))$ be a fundamental matrix of system (9.1) where each of $(n-m)$ columns of $\tilde{\Gamma}_\varphi(t)$ is a solution of system (9.1), then we have

$$\Psi(T,s) = \Phi(T)\Phi^{-1}(s), \quad \Gamma_\psi^*(0) = \Gamma_\psi^*(T) \tag{9.80}$$

and from Lemma 9.1.1 we have

$$\Gamma_\psi^*(T)\Phi(T) = \Gamma_\psi^*(s)\Phi(s). \tag{9.81}$$

From (9.79) and (9.80) we have

$$\Gamma_\psi^*(T) \left(\int_0^T \Phi(T)\Phi^{-1}(s)\boldsymbol{u}(s)ds + \sum_{0 \le \tau_k < T} \Phi(T)\Phi^{-1}(\tau_k^+)\boldsymbol{c}_k \right) = 0. \tag{9.82}$$

From (9.81) and (9.82) we have

$$\int_0^T \Gamma_\psi^*(s)\Phi(s)\Phi^{-1}(s)\boldsymbol{u}(s)ds + \sum_{0 \le \tau_k < T} \Gamma_\psi^*(\tau_k^+)\Phi(\tau_k^+)\Phi^{-1}(\tau_k^+)\boldsymbol{c}_k = 0 \tag{9.83}$$

from which we immediately have (9.68).
Conclusion 3: Assume that (9.68) holds and let us construct the following system:

$$\begin{aligned} (I - M)\boldsymbol{x}_T(0) &= \int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0 \le \tau_k < T} \Psi(T,\tau_k^+)\boldsymbol{c}_k, \\ \Gamma_\varphi^*(0)\boldsymbol{x}_T(0) &= 0. \end{aligned} \tag{9.84}$$

It follows from conclusion 3 of Lemma 9.1.2 that $P \triangleq I - M - \Gamma_\psi(0)\Gamma_\varphi^*(0)$ is nonsingular and system (9.84) has a unique solution

$$\boldsymbol{x}_T(0) = P^{-1}\boldsymbol{a} = P^{-1} \left(\int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0 \le \tau_k < T} \Psi(T,\tau_k^+)\boldsymbol{c}_k \right). \tag{9.85}$$

Therefore, system (9.46) has a unique T-periodic solution corresponding to $\boldsymbol{x}_T(0)$ as

$$\boldsymbol{x}_T(t) = \Psi(t,0)P^{-1}\left(\int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0\le\tau_k<T} \Psi(T,\tau_k^+)\boldsymbol{c}_k\right) + \int_0^t \Psi(t,s)\boldsymbol{u}(s)ds + \sum_{0\le\tau_k<t} \Psi(t,\tau_k^+)\boldsymbol{c}_k \quad (9.86)$$

which is exactly (9.71).

It follows from conclusion 3 of Lemma 9.1.2 that there is a constant $K > 0$ such that

$$\|\boldsymbol{x}_T(0)\| \le K \left\| \int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0\le\tau_k<T} \Psi(T,\tau_k^+)\boldsymbol{c}_k \right\| \quad (9.87)$$

from which and (9.86) we have estimation in (9.74). ■

Conclusions Based on Bounded Solutions.

Theorem 9.1.7. *If system (9.46) has a bounded solution then it has at least one T-periodic solution.* ⊠

Proof. Assume that $\boldsymbol{x}_1(t), t \ge 0$, is a bounded solution of system (9.46), then for $t \in \mathbb{R}_+$ we have

$$\boldsymbol{x}_1(t) = \Psi(t,0)\boldsymbol{x}_1(0) + \int_0^t \Psi(t,s)\boldsymbol{u}(s)ds + \sum_{0\le\tau_k<t} \Psi(t,\tau_k^+)\boldsymbol{c}_k \quad (9.88)$$

where $\Psi(t,s)$ is the Cauchy matrix of system (9.1). Let $M \triangleq \Psi(T,0)$ be a monodromy matrix of system (9.1) and set $\boldsymbol{a}$ as

$$\boldsymbol{a} \triangleq \int_0^T \Psi(T,s)\boldsymbol{u}(s)ds + \sum_{0\le\tau_k<T} \Psi(T,\tau_k^+)\boldsymbol{c}_k \quad (9.89)$$

then we have

$$\boldsymbol{x}_1(T) = M\boldsymbol{x}_1(0) + \boldsymbol{a}. \quad (9.90)$$

Since system (9.46) is T-periodic, we know that

$$\boldsymbol{x}_p(t) \triangleq \boldsymbol{x}_1(t+pT), \quad p \in \mathbb{N} \quad (9.91)$$

are also bounded solutions of system (9.46), then it follows from (9.90) that

$$\boldsymbol{x}_1(pT) = M^p\boldsymbol{x}_1(0) + \sum_{i=0}^{p-1} M^i\boldsymbol{a}. \quad (9.92)$$

We then have the following claim:

Claim 9.1.7: System (9.46) has at least one T-periodic solution.

If Claim 9.1.7 is not true, then system (9.46) has no T-periodic solution. Therefore, the equation

$$(I - M)\boldsymbol{x} = \boldsymbol{a} \tag{9.93}$$

has no solution. Then from the reverse of conclusion 2 of Lemma 9.1.2 we know that there is a $\boldsymbol{z} \in \mathbb{C}^n$ such that

$$(I - M^*)\boldsymbol{z} = 0 \text{ and } \boldsymbol{z}^*\boldsymbol{a} \neq 0, \tag{9.94}$$

from which we have

$$\boldsymbol{z} = M^*\boldsymbol{z}, \quad \boldsymbol{z} = (M^{-k})^*\boldsymbol{z} \text{ for } k \in \mathbb{N}. \tag{9.95}$$

Then from (9.92) we have

$$\boldsymbol{z}^*\boldsymbol{x}_1(pT) = \boldsymbol{z}^*M^p\boldsymbol{x}_1(0) + \sum_{i=0}^{p-1}\boldsymbol{z}^*M^i\boldsymbol{a}. \tag{9.96}$$

from which and (9.95) we have

$$\begin{aligned}\boldsymbol{z}^*\boldsymbol{x}_1(pT) &= [(M^{-p})^*\boldsymbol{z}]^*M^p\boldsymbol{x}_1(0) + \sum_{i=0}^{p-1}[(M^{-i})^*\boldsymbol{z}]^*M^i\boldsymbol{a} \\ &= \boldsymbol{z}^*M^{-p}M^p\boldsymbol{x}_1(0) + \sum_{i=0}^{p-1}\boldsymbol{z}^*M^{-i}M^i\boldsymbol{a} \\ &= \boldsymbol{z}^*\boldsymbol{x}_1(0) + p\boldsymbol{z}^*\boldsymbol{a}\end{aligned} \tag{9.97}$$

from which and (9.94) we have the following contradiction to the boundedness of $\boldsymbol{x}_1(t)$:

$$\lim_{p\to\infty}\boldsymbol{z}^*\boldsymbol{x}_1(pT) = \infty. \tag{9.98}$$

Therefore, Claim 9.1.7 is true.

■

We then have the following corollary.

Corollary 9.1.2.

1. *Assume that system (9.46) has no T-periodic solution, then all of its solutions are unbounded for $t \geq 0$ and $t \leq 0$.*
2. *Assume that system (9.46) has a unique bounded solution for $t \geq 0$, then this solution is T-periodic.*

⊠

First-order Cases. Let us then consider the following first-order linear impulsive T-periodic control system:

$$\begin{cases} \dot{x} = a(t)x + u(t), \, t = \tau_k, \\ \Delta x = b_k x + c_k, \quad t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \tag{9.99}$$

where $a \in \mathcal{PC}[\mathbb{R}, \mathbb{C}]$, $a(t+T) = a(t)$ for $t \in \mathbb{R}$, $u \in \mathcal{PC}[\mathbb{R}, \mathbb{C}]$, $u(t+T) = u(t)$, $b_k \in \mathbb{C}$, $1 + b_k \neq 0$, $c_k \in \mathbb{C}$, $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$. We assume that there is a $\varrho \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ we have

$$b_{k+\varrho} = b_k, \quad c_{k+\varrho} = c_k, \quad \tau_{k+\varrho} = \tau_k. \tag{9.100}$$

Then the Cauchy matrix for the following autonomous system

$$\begin{cases} \dot{x} = a(t)x, \; t = \tau_k, \\ \Delta x = b_k x, \; t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \tag{9.101}$$

is given by

$$\Psi(t,s) = \exp\left(\int_s^t a(\tau)d\tau\right) \prod_{s \le \tau_k < t} (1 + b_k), \quad -\infty < s \le t < \infty. \tag{9.102}$$

The multiplier of system (9.101) is given by

$$\mu = \Psi(\tau_\varrho^+, \tau_0^+) = \exp\left(\int_{\tau_0}^{\tau_\varrho} a(\tau)d\tau\right) \prod_{k=1}^{\varrho} (1 + b_k). \tag{9.103}$$

Then we have the following cases.

1. $\mu \neq 1$. In this case, system (9.99) has the following unique T-periodic solution:

$$\begin{aligned} x_T(t) = {} & \Psi(t, \tau_0^+) \frac{1}{1-\mu} \left(\int_{\tau_0}^{\tau_\varrho} \Psi(\tau_\varrho^+, s)u(s)ds + \sum_{k=1}^{\varrho} \Psi(\tau_\varrho^+, \tau_k^+)c_k \right) \\ & + \int_{\tau_0}^{t} \Psi(t,s)u(s)ds + \sum_{\tau_0 < \tau_k < t} \Psi(t, \tau_k^+)c_k. \end{aligned} \tag{9.104}$$

   It is easy to see that
   a) if $|\mu| < 1$, then $x_T(t)$ is exponentially stable;
   b) if $|\mu| > 1$, then $x_T(t)$ is unstable.
2. $\mu = 1$. In this case, all solution of system (9.101) are T-periodic, therefore all solutions of the following adjoint equation to (9.101) are also T-periodic:

$$\begin{cases} \dot{y} = -a^*(t)y, \quad t = \tau_k, \\ \Delta y = -\frac{b_k^*}{1+b_k^*} y, \; t = \tau_k, \quad k \in \mathbb{Z}. \end{cases} \tag{9.105}$$

And the function

$$y_1(t) = \exp\left(-\int_{\tau_0}^{t} a^*(\tau)d\tau\right) \prod_{\tau_0<\tau_k<t} \frac{1}{1+b_k^*} \tag{9.106}$$

is a T-periodic solution of system (9.105) with $y_1(\tau_\varrho^+) = y_1(\tau_0^+) = 1$. Furthermore, if the following condition holds:

$$\int_{\tau_0}^{\tau_\varrho} y_1^*(t)u(t)dt + \sum_{i=1}^{\varrho} y_1^*(\tau_i^+)c_i = 0 \tag{9.107}$$

then all solutions of (9.99) given by

$$x(t) = \Psi(t,\tau_0^+)x(\tau_0^+) + \int_{\tau_0}^{t} \Psi(t,s)u(s)ds + \sum_{\tau_0<\tau_k<t} \Psi(t,\tau_k^+)c_k \tag{9.108}$$

are T-periodic and stable.

3. $\varrho = 1$, $b_k = b$, $c_k = c$ for $k \in \mathbb{Z}$. In this case, (9.103) can be simplified as

$$\mu = \exp\left(\int_{\tau_0}^{\tau_1} a(\tau)d\tau\right)(1+b). \tag{9.109}$$

If we further assume that $\mu = 1$, then (9.107) can be simplified as

$$\int_{\tau_0}^{\tau_1} \exp\left(-\int_{\tau_0}^{t} a(\tau)d\tau\right) u(t)dt + c = 0. \tag{9.110}$$

9.2 Parameter Perturbation Methods and Robustness

In this section we provide theoretical basis for two kinds of impulsive control strategies for controlling periodic motions; namely,

1. Given a periodic motion, how can we design impulsive periodic motion controllers that are robust to parameter drift?
2. How can we use parameter perturbation methods to design impulsive periodic motion controllers?

9.2.1 Linear Control Systems

Let us study the following impulsive control system with parameter perturbations:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t) + \boldsymbol{p}(t, \boldsymbol{x}, \xi),\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x} + \boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}, \xi), \quad t = \tau_k, \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}, \end{cases} \tag{9.111}$$

where $\boldsymbol{x} \in \mathcal{S}_\rho \subset \mathbb{R}^n$, $\xi \in \beth \triangleq (-\tilde{\xi}, \tilde{\xi})$ is a small parameter perturbation that may be caused by some adaptive impulsive control algorithms or parameter drift.

$A \in \mathcal{PC}[\mathbb{R}, \mathbb{R}^{n\times n}]$, $A(t+T) = A(t)$ for $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathcal{PC}[\mathbb{R}, \mathbb{R}^n]$, $\boldsymbol{u}(t+T) = \boldsymbol{u}(t)$, $B_k \in \mathbb{R}^{n\times n}$, $\det(I + B_k) \neq 0$, $\boldsymbol{c}_k \in \mathbb{R}^n$, $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$. We assume that there is a $\varrho \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ we have

$$B_{k+\varrho} = B_k, \quad \boldsymbol{c}_{k+\varrho} = \boldsymbol{c}_k, \quad \tau_{k+\varrho} = \tau_k + T. \tag{9.112}$$

$\boldsymbol{p} \in C[(\tau_k, \tau_{k+1}] \times \mathcal{S}_\rho \times \beth, \mathbb{R}^n], k \in \mathbb{Z}$, for any $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$ and $\xi \in \beth$, we have

$$\lim_{(t,\boldsymbol{y},\varepsilon)\to(\tau_k^+,\boldsymbol{x},\xi)} \boldsymbol{p}(t, \boldsymbol{y}, \varepsilon) < \infty, \text{ and } \boldsymbol{p}(t+T, \boldsymbol{x}, \xi) = \boldsymbol{p}(t, \boldsymbol{x}, \xi).$$

$\boldsymbol{q}_k : C[\mathcal{S}_\rho \times \beth, \mathbb{R}^n]$ and

$$\boldsymbol{q}_{k+\varrho}(\boldsymbol{x}, \xi) = \boldsymbol{q}_k(\boldsymbol{x}, \xi),$$

for all $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$, and $\xi \in \beth$. Let χ be a nonnegative function such that for $t \in \mathbb{R}$, $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$, and $\xi \in \beth$,

$$\lim_{\xi\to 0} \chi(\xi) = \chi(0) = 0$$

and

$$\|\boldsymbol{p}(t, \boldsymbol{x}, \xi)\| \leq \chi(\xi), \quad \|\boldsymbol{q}_k(\boldsymbol{x}, \xi)\| \leq \chi(\xi). \tag{9.113}$$

Let us suppose that the following reference system has no T-periodic solution:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x},\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x},\ t = \tau_k, \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}. \end{cases} \tag{9.114}$$

Let $\xi = 0$, then system (9.111) has the same form as system (9.46) because it follows from (9.113) that $\boldsymbol{p}(t, \boldsymbol{x}, 0) = 0$ and $\boldsymbol{q}_k(\boldsymbol{x}, 0) = 0$. It follows from Theorem 9.1.5 that system (9.111) has the following T-periodic solution when $\xi = 0$:

$$\boldsymbol{x}_T(t) = \int_0^T \Upsilon(t, s)\boldsymbol{u}(s)ds + \sum_{k=1}^{\varrho} \Upsilon(t, \tau_k^+)\boldsymbol{c}_k, \tag{9.115}$$

where $\Upsilon(t, s)$ is the Green's function defined in (9.50). Then we have the following theorem to provide sufficient conditions for the existence of T-periodic solutions of system (9.111).

Theorem 9.2.1. *Assume that*

1.
$$\rho_0 = \sup_{t\in[0,T]} \left\| \int_0^T \Upsilon(t,s)\boldsymbol{u}(s)ds + \sum_{k=1}^{\varrho} \Upsilon(t,\tau_k^+)\boldsymbol{c}_k \right\| < \rho; \tag{9.116}$$

2. *System (9.114) has no T-periodic solution;*
3. *Let ϖ be a nonnegative function such that for $k \in \mathbb{Z}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_\rho$, and $\xi \in \beth$,*
$$\lim_{\xi\to 0} \varpi(\xi) = \varpi(0) = 0$$
and
$$\|\boldsymbol{p}(t,\boldsymbol{x},\xi)-\boldsymbol{p}(t,\boldsymbol{y},\xi)\| \le \varpi(\xi)\|\boldsymbol{x}-\boldsymbol{y}\|, \;\; \|\boldsymbol{q}_k(\boldsymbol{x},\xi)-\boldsymbol{q}_k(\boldsymbol{y},\xi)\| \le \varpi(\xi)\|\boldsymbol{x}-\boldsymbol{y}\|.$$

Then, there is a $\xi_0 \in (0,\tilde{\xi})$ such that for $|\xi| \le \xi_0$ system (9.111) has a unique T-period solution $\boldsymbol{x}_T^{\xi}(t)$ satisfying

$$\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\| < \rho - \rho_0 \tag{9.117}$$

and

$$\lim_{\xi\to 0} \boldsymbol{x}_T^{\xi}(t) = \boldsymbol{x}_T(t) \tag{9.118}$$

uniformly on $t \in \mathbb{R}$. ⊠

Proof. Let $\mho$ be the Banach space of T-periodic solutions $\boldsymbol{x} \in \mathcal{PC}[\mathbb{R},\mathbb{R}^n]$ with norm
$$\|\boldsymbol{x}\|_\mho = \sup_{t\in[0,T]} \|\boldsymbol{x}(t)\|.$$

Let us define

$$\begin{aligned} \rho_1 &= \rho - \rho_0, \\ \mho(\boldsymbol{x}_T,\rho_1) &\triangleq \{\boldsymbol{x} \in \mho \mid \|\boldsymbol{x}-\boldsymbol{x}_T\|_\mho \le \rho_1\}, \\ L &= \sup_{t,s\in[0,T]} \|\Upsilon(t,s)\| \end{aligned} \tag{9.119}$$

and an operator $\mathcal{O} : \mho(\boldsymbol{x}_T,\rho_1) \to \mho$ as

$$\begin{aligned} \mathcal{O}(\boldsymbol{x}) \triangleq &\int_0^T \Upsilon(t,s)[\boldsymbol{u}(s) + \boldsymbol{p}(s,\boldsymbol{x}(s),\xi)]ds \\ &+ \sum_{k=1}^{\varrho} \Upsilon(t,\tau_k^+)[\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}(\tau_k),\xi)]. \end{aligned} \tag{9.120}$$

From (9.115), (9.116) and (9.119) we know that if $\boldsymbol{x} \in \mho(\boldsymbol{x}_T,\rho_1)$ then

$$\begin{aligned}\|\boldsymbol{x}(t)\| &\leq \|\boldsymbol{x}(t)-\boldsymbol{x}_T(t)\|+\|\boldsymbol{x}_T(t)\| \\ &\leq \rho_1+\sup_{t\in[0,T]}\|\boldsymbol{x}_T(t)\| \qquad \Leftarrow (9.119) \\ &= \rho_1+\sup_{t\in[0,T]}\left\|\int_0^T \Upsilon(t,s)\boldsymbol{u}(s)ds+\sum_{k=1}^{\varrho}\Upsilon(t,\tau_k^+)\boldsymbol{c}_k\right\| \qquad \Leftarrow (9.115) \\ &= \rho_1+\rho_0=\rho \qquad \Leftarrow (9.116).\end{aligned} \tag{9.121}$$

Given $\boldsymbol{x},\boldsymbol{y}\in\mho(\boldsymbol{x}_T,\rho_1)$ then we have

$$\begin{aligned}\|\mathcal{O}(\boldsymbol{x})-\mathcal{O}(\boldsymbol{y})\|_{\mho} &= \sup_{t\in[0,T]}\left\|\int_0^T \Upsilon(t,s)[\boldsymbol{p}(s,\boldsymbol{x}(s),\xi)-\boldsymbol{p}(s,\boldsymbol{y}(s),\xi)]ds\right. \\ &\quad \left.+\sum_{k=1}^{\varrho}\Upsilon(t,\tau_k^+)[\boldsymbol{q}_k(\boldsymbol{x}(\tau_k),\xi)-\boldsymbol{q}_k(\boldsymbol{y}(\tau_k),\xi)]\right\| \\ &\leq L(T+\varrho)\varpi(\xi)\|\boldsymbol{x}-\boldsymbol{y}\|_{\mho},\end{aligned} \tag{9.122}$$

and

$$\begin{aligned}\|\mathcal{O}(\boldsymbol{x}_T)-\boldsymbol{x}_T\|_{\mho} &= \sup_{t\in[0,T]}\left\|\int_0^T \Upsilon(t,s)\boldsymbol{p}(s,\boldsymbol{x}_T(s),\xi)ds\right. \\ &\quad \left.+\sum_{k=1}^{\varrho}\Upsilon(t,\tau_k^+)\boldsymbol{q}_k(\boldsymbol{x}_T(\tau_k),\xi)\right\| \\ &\leq L(T+\varrho)\chi(\xi).\end{aligned} \tag{9.123}$$

Let us choose a $\xi_0\in(0,\tilde{\xi})$ such that

$$\begin{aligned}&\eta=L(T+\varrho)\sup_{|\xi|\leq\xi_0}\varpi(\xi)<1, \\ &L(T+\varrho)\sup_{|\xi|\leq\xi_0}\chi(\xi)\leq\rho_1(1-\eta).\end{aligned} \tag{9.124}$$

Assume that $|\xi|\leq\xi_0$, then it follows from (9.122), (9.123) and (9.124) that

$$\begin{aligned}&\|\mathcal{O}(\boldsymbol{x})-\mathcal{O}(\boldsymbol{y})\|_{\mho}\leq\eta\|\boldsymbol{x}-\boldsymbol{y}\|_{\mho}, \\ &\|\mathcal{O}(\boldsymbol{x})-\boldsymbol{x}_T\|_{\mho}\leq\rho_1(1-\eta).\end{aligned} \tag{9.125}$$

Then from Banach's fixed point theorem we know that the operator $\mathcal{O}$ has a unique fixed point $\boldsymbol{x}_T^{\xi}\in\mho(\boldsymbol{x}_T,\rho_1)$ satisfying

$$\begin{aligned}\boldsymbol{x}_T^{\xi}(t) &= \int_0^T \Upsilon(t,s)[\boldsymbol{u}(s)+\boldsymbol{p}(s,\boldsymbol{x}_T^{\xi}(s),\xi)]ds \\ &\quad +\sum_{k=1}^{\varrho}\Upsilon(t,\tau_k^+)[\boldsymbol{c}_k+\boldsymbol{q}_k(\boldsymbol{x}_T^{\xi}(\tau_k),\xi)].\end{aligned} \tag{9.126}$$

It is clear that $\boldsymbol{x}_T^{\xi}(t)$ is a T-periodic solution of system (9.111) and satisfies estimate (9.117) because

$$\begin{aligned}
&\boldsymbol{x}_T^{\xi}(t) \in \mho(\boldsymbol{x}_T, \rho_1) \\
\Rightarrow &\left\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\right\|_{\mho} \leq \rho_1 \\
\Rightarrow &\sup_{t\in[0,T]} \left\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\right\| \leq \rho_1 \\
\Rightarrow &\left\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\right\| \leq \rho_1 = \rho - \rho_0.
\end{aligned} \tag{9.127}$$

$\boldsymbol{x}_T^{\xi}(t)$ is a limit of a uniformly convergent sequence of T-periodic functions satisfying

$$\begin{aligned}
\boldsymbol{x}_i(t) = &\int_0^T \Upsilon(t,s)[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}_{i-1}(s), \xi)]ds \\
&+ \sum_{k=1}^{\varrho} \Upsilon(t, \tau_k^+)[\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}_{i-1}(\tau_k), \xi)], \quad i \in \mathbb{N}
\end{aligned} \tag{9.128}$$

from which we know that $\boldsymbol{x}_i(t) = \mathcal{O}(\boldsymbol{x}_{i-1}(t))$. And we also know that $\boldsymbol{x}_T^{\xi}(t) = \mathcal{O}(\boldsymbol{x}_T^{\xi}(t))$, then we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left\|\boldsymbol{x}_{i+1}(t) - \boldsymbol{x}_T^{\xi}(t)\right\| \\
&= \left\|\boldsymbol{x}_{i+1}(t) - \boldsymbol{x}_T^{\xi}(t)\right\|_{\mho} \\
&= \left\|\mathcal{O}(\boldsymbol{x}_i(t)) - \mathcal{O}(\boldsymbol{x}_T^{\xi}(t))\right\|_{\mho} \\
&\leq \eta \left\|\boldsymbol{x}_i(t) - \boldsymbol{x}_T^{\xi}(t)\right\|_{\mho} \qquad \Leftarrow \text{[from the first inequality in (9.125)]} \\
&= \left\|\mathcal{O}(\boldsymbol{x}_{i-1}(t)) - \mathcal{O}(\boldsymbol{x}_T^{\xi}(t))\right\|_{\mho} \\
&\leq \cdots \\
&\leq \eta^i \left\|\boldsymbol{x}_1(t) - \boldsymbol{x}_T^{\xi}(t)\right\|_{\mho} \\
&\leq \eta^i \left(\|\boldsymbol{x}_1(t) - \boldsymbol{x}_T(t)\|_{\mho} + \left\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\right\|_{\mho}\right) \\
&\leq 2\eta^i \rho_1(1-\eta) \quad \text{provided } |\xi| \leq \xi_0, \quad i \in \mathbb{N}. \\
&\Uparrow \text{[from the second inequality in (9.125)]}
\end{aligned} \tag{9.129}$$

Then we have the following estimate:

$$\begin{aligned}\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\| &= \sup_{t\in\mathbb{R}} \left\| \int_0^T \Upsilon(t,s)\boldsymbol{p}(s,\boldsymbol{x}_T^{\xi}(s),\xi)ds \right. \\ &\quad \left. + \sum_{k=1}^{\varrho} \Upsilon(t,\tau_k^+)\boldsymbol{q}_k(\boldsymbol{x}_T^{\xi}(\tau_k),\xi) \right\| \\ &\le L(T+\varrho)\chi(\xi) \end{aligned} \tag{9.130}$$

from which and (9.129) we have the conclusion (9.118). ■

The following definition and lemma will be used in the proof of Theorem 9.2.2.

Definition 9.2.1. *A set* Ⓢ $\subset \mathcal{PC}[[0,T],\mathbb{R}^n]$ *is quasiequicontinuous in* $[0,T]$ *if for any* $\delta > 0$ *there is a* $\varepsilon > 0$ *such that if* $\boldsymbol{x} \in$ Ⓢ, $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [0,T]$, $k \in \mathbb{Z}$, *and* $|t_1 - t_2| < \varepsilon$, *then* $\|\boldsymbol{x}(t_1) - \boldsymbol{x}(t_2)\| < \delta$. ⊠

Lemma 9.2.1. *A set* Ⓢ $\subset \mathcal{PC}[[0,T],\mathbb{R}^n]$ *is* relatively compact *if and only if*

1. Ⓢ *is bounded for each* $\boldsymbol{x} \in$ Ⓢ*;*
2. Ⓢ *is quasiequicontinuous in* $[0,T]$.

⊠

Theorem 9.2.2. *Assume that*

1. $$\rho_0 = \sup_{t\in[0,T]} \left\| \int_0^T \Upsilon(t,s)\boldsymbol{u}(s)ds + \sum_{k=1}^{\varrho} \Upsilon(t,\tau_k^+)\boldsymbol{c}_k \right\| < \rho; \tag{9.131}$$
2. *System (9.114) has no* T*-periodic solution.*

Then, there is a $\xi_0 \in (0,\tilde{\xi})$ *such that for* $|\xi| \le \xi_0$ *system (9.111) has a unique* T*-period solution* $\boldsymbol{x}_T^{\xi}(t)$ *satisfying*

$$\|\boldsymbol{x}_T^{\xi}(t) - \boldsymbol{x}_T(t)\| < \rho - \rho_0. \tag{9.132}$$

⊠

Proof. Let $\mho$ be the Banach space of T-periodic solutions $\boldsymbol{x} \in \mathcal{PC}[\mathbb{R},\mathbb{R}^n]$ with norm

$$\|\boldsymbol{x}\|_{\mho} = \sup_{t\in[0,T]} \|\boldsymbol{x}(t)\|.$$

Let us define

$$\begin{aligned} \rho_1 &= \rho - \rho_0, \\ \mho(\boldsymbol{x}_T,\rho_1) &\triangleq \{\boldsymbol{x} \in \mho \mid \|\boldsymbol{x} - \boldsymbol{x}_T\|_{\mho} \le \rho_1\}, \\ L &= \sup_{t,s\in[0,T]} \|\Upsilon(t,s)\| \end{aligned} \tag{9.133}$$

and an operator $\mathcal{O} : \mho(\boldsymbol{x}_T, \rho_1) \to \mho$ as

$$\mathcal{O}(\boldsymbol{x}) \triangleq \int_0^T \Upsilon(t,s)[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]ds \\ + \sum_{k=1}^{\varrho} \Upsilon(t, \tau_k^+)[\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}(\tau_k), \xi)]. \tag{9.134}$$

We know that $\mho(\boldsymbol{x}_T, \rho_1)$ is nonempty, bounded, closed and convex. It follows from the condition in (9.121) that if $\boldsymbol{x}(t) \in \mho(\boldsymbol{x}_T, \rho_1)$ then $\|\boldsymbol{x}(t)\| \le \rho$ for all $t \in \mathbb{R}$. Let us choose $\xi_0 \in (0, \tilde{\xi})$ such that

$$L(T + \varrho) \sup_{|\xi| \le \xi_0} \chi(\xi) \le \rho_1 \tag{9.135}$$

then we have

$$\begin{aligned} \|\mathcal{O}(\boldsymbol{x}) - \boldsymbol{x}_T\|_\mho = \sup_{t\in[0,T]} &\left\| \int_0^T \Upsilon(t,s)\boldsymbol{p}(s, \boldsymbol{x}(s), \xi)ds \right. \\ &\left. + \sum_{k=1}^{\varrho} \Upsilon(t, \tau_k^+)\boldsymbol{q}_k(\boldsymbol{x}(\tau_k), \xi) \right\| \\ &\le L(T + \varrho)\chi(\xi) \le \rho_1, \text{ provided } |\xi| \le \xi_0, \end{aligned} \tag{9.136}$$

from which we know that $\mathcal{O}(\boldsymbol{x}) \in \mho(\boldsymbol{x}_T, \rho_1)$ and therefore $\mathcal{O}(\boldsymbol{x}) : \mho(\boldsymbol{x}_T, \rho_1) \to \mho(\boldsymbol{x}_T, \rho_1)$. It follows from (9.115), (9.131) and (9.136) that

$$\|\mathcal{O}(\boldsymbol{x})\|_\mho \le \|\mathcal{O}(\boldsymbol{x}) - \boldsymbol{x}_T\|_\mho + \|\boldsymbol{x}_T\|_\mho \le \rho_1 + \rho_0 = \rho \tag{9.137}$$

from which we know that $\mho(\boldsymbol{x}_T, \rho_1)$ is uniformly bounded.

Let us set $\boldsymbol{x} \in \mho(\boldsymbol{x}_T, \rho_1)$, $t_1, t_2 \in (\tau_{m-1}, \tau_m] \cap [0, T]$, $m = 1, \cdots, \varrho + 1$ and $t_1 < t_2$, then from the definition of $\Upsilon(t, s)$ in (9.50) we have

$$
\begin{aligned}
&\|\mathcal{O}(\boldsymbol{x}(t_1)) - \mathcal{O}(\boldsymbol{x}(t_2))\| \\
\le &\left\| \int_0^{t_1} [\Upsilon(t_1, s) - \Upsilon(t_2, s)][\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]ds \right\| \\
&+ \left\| \int_{t_2}^{T} [\Upsilon(t_1, s) - \Upsilon(t_2, s)][\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]ds \right\| \\
&+ \int_{t_1}^{t_2} (\|\Upsilon(t_1, s)\| + \|\Upsilon(t_2, s)\|) \, \|[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]\| ds \\
&+ \left\| \sum_{k=1}^{\varrho} [\Upsilon(t_1, \tau_k^+) - \Upsilon(t_2, \tau_k^+)][\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}(\tau_k), \xi)] \right\| \\
= &\left\| \int_0^{t_1} [X(t_1) - X(t_2)][I - X(T)]^{-1} X^{-1}(s)[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]ds \right\| \\
&+ \Big\| \int_{t_2}^{T} [X(t_1 + T) - X(t_2 + T)][I - X(T)]^{-1} X^{-1}(s) \\
&\times [\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]ds \Big\| \\
&+ \int_{t_1}^{t_2} (\|\Upsilon(t_1, s)\| + \|\Upsilon(t_2, s)\|) \, \|[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}(s), \xi)]\| ds \\
&+ \Big\| \sum_{k=1}^{m-1} [X(t_1) - X(t_2)][I - X(T)]^{-1} X^{-1}(\tau_k^+)] \\
&\times [\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}(\tau_k), \xi)] \Big\| \\
&+ \Big\| \sum_{k=m}^{\varrho} [X(t_1 + T) - X(t_2 + T)][I - X(T)]^{-1} X^{-1}(\tau_k^+)] \\
&\times [\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}(\tau_k), \xi)] \Big\|
\end{aligned} \tag{9.138}
$$

from which we know that for any $\delta > 0$, there is a $\varepsilon > 0$ such that if $(t_2 - t_1) < \varepsilon$, then $\|\mathcal{O}(\boldsymbol{x}(t_1)) - \mathcal{O}(\boldsymbol{x}(t_2))\| < \delta$. Thus, it follows from Definition 9.2.1 that $\mho(\boldsymbol{x}_T, \rho_1)$ is quasiequicontinuous. Then from Lemma 9.2.1 we know that the following set is relatively compact in $\mho$:

$$
Ⓢ = \{\boldsymbol{y} \in \mho \mid \boldsymbol{y} = \mathcal{O}(\boldsymbol{x}), \boldsymbol{x} \in \mho(\boldsymbol{x}_T, \rho_1)\}. \tag{9.139}
$$

Therefore, the operator $\mathcal{O}(\boldsymbol{x})$ has a fixed point $\boldsymbol{x}_T^{\xi}(t) \in \mho(\boldsymbol{x}_T, \rho_1)$ that is T-periodic and satisfies

$$
\begin{aligned}
\boldsymbol{x}_T^{\xi}(t) = &\int_0^T \Upsilon(t, s)[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}_T^{\xi}(s), \xi)]ds \\
&+ \sum_{k=1}^{\varrho} \Upsilon(t, \tau_k^+)[\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}_T^{\xi}(\tau_k), \xi)].
\end{aligned} \tag{9.140}
$$

It is clear that $\boldsymbol{x}_T^{\xi}(t)$ is a T-periodic solution of system (9.111). ■

9.2.2 Nonlinear Control Systems

Let us study the following nonlinear impulsive periodic motion control system with parameter perturbations:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{u}(t) + \boldsymbol{p}(t,\boldsymbol{x},\xi) + \boldsymbol{v}(t,\boldsymbol{x}), & t \neq \tau_k, \\ \Delta \boldsymbol{x} = U(k,\boldsymbol{x}) + \boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x},\xi) + \boldsymbol{v}_k(t,\boldsymbol{x}), t = \tau_k, & t \in \mathbb{R}, \quad k \in \mathbb{Z}, \end{cases} \qquad (9.141)}$$

where $\boldsymbol{x} \in \mathcal{S}_\rho \subset \mathbb{R}^n$, $\xi \in \beth \triangleq (-\tilde{\xi},\tilde{\xi})$ is a small parameter perturbation that may be caused by some adaptive impulsive control algorithms or parameter drift.

$\boldsymbol{f} \in C^1[(\tau_k,\tau_{k+1}] \times \mathcal{S}_\rho, \mathbb{R}^n]$ for all $k \in \mathbb{Z}$ and

$$\lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \boldsymbol{f}(t,\boldsymbol{y}) < \infty, \lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \frac{\partial \boldsymbol{f}(t,\boldsymbol{y})}{\partial \boldsymbol{x}} < \infty, \boldsymbol{f}(t+T,\boldsymbol{x}) = \boldsymbol{f}(t,\boldsymbol{x})$$

for $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$ and $t \in \mathbb{R}$.

$\boldsymbol{u} \in \mathcal{PC}[\mathbb{R},\mathbb{R}^n]$, $\boldsymbol{u}(t+T) = \boldsymbol{u}(t)$. $\boldsymbol{p} \in C[(\tau_k,\tau_{k+1}] \times \mathcal{S}_\rho \times \beth, \mathbb{R}^n], k \in \mathbb{Z}$, for any $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$ and $\xi \in \beth$, we have

$$\lim_{(t,\boldsymbol{y},\varepsilon)\to(\tau_k^+,\boldsymbol{x},\xi)} \boldsymbol{p}(t,\boldsymbol{y},\varepsilon) < \infty, \text{ and } \boldsymbol{p}(t+T,\boldsymbol{x},\xi) = \boldsymbol{p}(t,\boldsymbol{x},\xi).$$

$\boldsymbol{v} \in C[(\tau_k,\tau_{k+1}] \times \mathcal{S}_\rho, \mathbb{R}^n]$ and

$$\lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \boldsymbol{v}(t,\boldsymbol{y}) < \infty, \quad k \in \mathbb{Z}, \quad \boldsymbol{x} \in \mathcal{S}_\rho.$$

$$\boldsymbol{v}(t+T,\boldsymbol{x}) = \boldsymbol{v}(t,\boldsymbol{x}), t \in \mathbb{R}, \boldsymbol{x} \in \mathcal{S}_\rho$$

$U_k \in C^1[\mathbb{Z} \times \mathcal{S}_\rho, \mathbb{R}^n]$ and

$$U(k+\varrho,\boldsymbol{x}) = U(k,\boldsymbol{x}) \text{ for } k \in \mathbb{Z}, \boldsymbol{x} \in \mathcal{S}_\rho.$$

$\boldsymbol{c}_k \in \mathbb{R}^n$, $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$. We assume that there is a $\varrho \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ we have

$$\boldsymbol{c}_{k+\varrho} = \boldsymbol{c}_k, \quad \tau_{k+\varrho} = \tau_k + T. \qquad (9.142)$$

$\boldsymbol{q}_k : C[\mathcal{S}_\rho \times \beth, \mathbb{R}^n]$ and

$$\boldsymbol{q}_{k+\varrho}(\boldsymbol{x},\xi) = \boldsymbol{q}_k(\boldsymbol{x},\xi),$$

for all $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$, and $\xi \in \beth$.

$\boldsymbol{v}_k \in C[\mathcal{S}_\rho, \mathbb{R}^n]$ and

$$\boldsymbol{v}_{k+\varrho}(\boldsymbol{x}) = \boldsymbol{v}_k(\boldsymbol{x}), \quad k \in \mathbb{Z}, \quad \boldsymbol{x} \in \mathcal{S}_\rho.$$

Let χ be a nonnegative function such that for $t \in \mathbb{R}$, $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathcal{S}_\rho$, and $\xi \in \beth$,

$$\lim_{\xi \to 0} \chi(\xi) = \chi(0) = 0$$

and

$$\|\boldsymbol{p}(t, \boldsymbol{x}, \xi)\| \leq \chi(\xi), \quad \|\boldsymbol{q}_k(\boldsymbol{x}, \xi)\| \leq \chi(\xi).$$

Let ϖ be a nonnegative function such that for $k \in \mathbb{Z}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_\rho$, and $\xi \in \beth$,

$$\lim_{\xi \to 0} \varpi(\xi) = \varpi(0) = 0$$

and

$$\|\boldsymbol{p}(t, \boldsymbol{x}, \xi) - \boldsymbol{p}(t, \boldsymbol{y}, \xi)\| \leq \varpi(\xi)\|\boldsymbol{x} - \boldsymbol{y}\|, \ \|\boldsymbol{q}_k(\boldsymbol{x}, \xi) - \boldsymbol{q}_k(\boldsymbol{y}, \xi)\| \leq \varpi(\xi)\|\boldsymbol{x} - \boldsymbol{y}\|.$$

Assume that $\boldsymbol{x}_T(t)$ is a T-periodic solution of the following reference system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), & t \neq \tau_k, \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), & t = \tau_k, \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}, \end{cases} \tag{9.143}$$

and

$$\boldsymbol{v}(t, \boldsymbol{x}_T(t)) = 0, \quad \boldsymbol{v}_k(\boldsymbol{x}_T(\tau_k)) = 0, \quad t \in \mathbb{R}, \quad k \in \mathbb{Z},$$

$$\|\boldsymbol{v}(t, \boldsymbol{x}) - \boldsymbol{v}(t, \boldsymbol{y})\| \leq K_1(\varepsilon)\|\boldsymbol{x} - \boldsymbol{y}\|, \quad t \in \mathbb{R}, \|\boldsymbol{x} - \boldsymbol{x}_T(t)\| \leq \varepsilon, \|\boldsymbol{y} - \boldsymbol{x}_T(t)\| \leq \varepsilon,$$

where $K_1(\varepsilon) > 0$ and

$$\lim_{\varepsilon \to 0} K_1(\varepsilon) = 0.$$

$$\|\boldsymbol{v}_k(\boldsymbol{x}) - \boldsymbol{v}_k(\boldsymbol{y})\| \leq K_2(\varepsilon)\|\boldsymbol{x} - \boldsymbol{y}\|, \quad k \in \mathbb{Z}, \|\boldsymbol{x} - \boldsymbol{x}_T(\tau_k)\| \leq \varepsilon, \|\boldsymbol{y} - \boldsymbol{x}_T(\tau_k)\| \leq \varepsilon,$$

where $K_2(\varepsilon) > 0$ and

$$\lim_{\varepsilon \to 0} K_2(\varepsilon) = 0.$$

Let $\Upsilon(t, s)$ be the Green's function of the following T-periodic linear impulsive system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t))}{\partial \boldsymbol{x}} \boldsymbol{x}, \quad t \neq \tau_k, \\ \Delta \boldsymbol{x} &= \frac{\partial U(k, \boldsymbol{x}_T(t))}{\partial \boldsymbol{x}} \boldsymbol{x}, \quad t = \tau_k. \end{aligned} \tag{9.144}$$

Theorem 9.2.3. *Assume that*

1. *The T-periodic solution, $\boldsymbol{x}_T(t)$, of the reference system (9.143) satisfies $\|\boldsymbol{x}_T(t)\| \leq \rho_0 < \rho, t \in \mathbb{R}$;*

2. *System (9.144) does not have nontrivial T-periodic solution;*
3. *Let $\xi_0 \in (0, \tilde{\xi})$ and $\varepsilon_0 \in (0, \rho - \rho_0)$, such that $\eta < 1$ with*

$$\eta \triangleq L \left[TK_1(\varepsilon_0) + \varrho K_2(\varepsilon_0) + TK_3(\varepsilon_0) + \varrho K_4(\varepsilon_0) + (T + \varrho) \sup_{|\xi|<\xi_0} \varpi(\xi) \right]$$

where

$$\begin{aligned} L &= \sup_{t,s\in[0,T]} \|\Upsilon(t,s)\|, \\ K_3(\varepsilon) &= \sup_{t\in[0,T],\|\boldsymbol{y}\|\le\varepsilon} \left\| \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t) + \boldsymbol{y})}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t))}{\partial \boldsymbol{x}} \right\|, \\ K_4(\varepsilon) &= \sup_{k\in\mathbb{Z},\|\boldsymbol{y}\|\le\varepsilon} \left\| \frac{\partial U(k, \boldsymbol{x}_T(\tau_k) + \boldsymbol{y})}{\partial \boldsymbol{x}} - \frac{\partial U(k, \boldsymbol{x}_T(\tau_k))}{\partial \boldsymbol{x}} \right\|. \end{aligned}$$

4.

$$\begin{aligned} \sup_{t\in[0,T],|\xi|\le\xi_0} & \left\| \int_0^T \Upsilon(t,s)[\boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}_T(s), \xi)]ds \right. \\ & \left. + \sum_{0\le\tau_k<T} \Upsilon(t, \tau_k^+)[\boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}_T(\tau_k), \xi)] \right\| < \varepsilon_0(1-\eta). \end{aligned}$$

Then for each $\xi \in (-\xi_0, \xi_0)$ system (9.141) has a unique T-periodic solution $\boldsymbol{x}_T^\xi(t)$ satisfying

$$\|\boldsymbol{x}_T^\xi(t) - \boldsymbol{x}_T(t)\| \le \varepsilon_0 \text{ for } t \in \mathbb{R}, \tag{9.145}$$

$$\lim_{\xi\to 0} \boldsymbol{x}_T^\xi(t) = \boldsymbol{x}_T(t) \text{ uniformly on } t \in \mathbb{R}. \tag{9.146}$$

⊠

Proof. Let $\boldsymbol{x} = \boldsymbol{x}_T(t) + \boldsymbol{y}$, then we can change system (9.141) into

$$\begin{aligned} \dot{\boldsymbol{y}} &= \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t))}{\partial \boldsymbol{x}} \boldsymbol{y} + \boldsymbol{o}(t, \boldsymbol{y}) + \boldsymbol{u}(t) + \boldsymbol{p}(t, \boldsymbol{x}_T(t) + \boldsymbol{y}, \xi) \\ &\quad + \boldsymbol{v}(t, \boldsymbol{x}_T(t) + \boldsymbol{y}), \quad t \ne \tau_k, \\ \Delta \boldsymbol{y} &= \frac{\partial U(k, \boldsymbol{x}_T(\tau_k))}{\partial \boldsymbol{x}} \boldsymbol{y} + \boldsymbol{o}_k(\boldsymbol{y}) + \boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}_T(\tau_k) + \boldsymbol{y}, \xi) \\ &\quad + \boldsymbol{v}_k(\boldsymbol{x}_T(\tau_k) + \boldsymbol{y}), \quad t = \tau_k, \end{aligned} \tag{9.147}$$

where

$$\begin{aligned} \boldsymbol{o}(t, \boldsymbol{y}) &= \boldsymbol{f}(t, \boldsymbol{x}_T(t) + \boldsymbol{y}) - \boldsymbol{f}(t, \boldsymbol{x}_T(t)) - \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t))}{\partial \boldsymbol{x}} \boldsymbol{y}, \\ \boldsymbol{o}_k(\boldsymbol{y}) &= U_k(\boldsymbol{x}_T(\tau_k) + \boldsymbol{y}) - U_k(\boldsymbol{x}_T(\tau_k)) - \frac{\partial U_k(\boldsymbol{x}_T(\tau_k))}{\partial \boldsymbol{x}} \boldsymbol{y}. \end{aligned} \tag{9.148}$$

Let $\mho$ be the Banach space of T-periodic solutions $\boldsymbol{x} \in \mathcal{PC}[\mathbb{R}, \mathbb{R}^n]$ with norm

$$\|\boldsymbol{x}\|_{\mho} = \sup_{t\in[0,T]} \|\boldsymbol{x}(t)\|.$$

Let us define $\mho(\varepsilon_0) \triangleq \{\boldsymbol{y} \in \mho \mid \|\boldsymbol{y}\|_{\mho} \leq \varepsilon_0\}$ and the operator $\mathcal{O} : \mho(\varepsilon_0) \to \mho$ as

$$\begin{aligned}
\mathcal{O}(\boldsymbol{x}) \triangleq & \int_0^T \Upsilon(t,s)[\boldsymbol{o}(s,\boldsymbol{y}(s)) + \boldsymbol{u}(s) + \boldsymbol{p}(s, \boldsymbol{x}_T(s) + \boldsymbol{y}(s), \xi) \\
& + \boldsymbol{v}(s, \boldsymbol{x}_T(s)) + \boldsymbol{y}(s))]ds \\
& + \sum_{0\leq\tau_k<T} \Upsilon(t, \tau_k^+)[\boldsymbol{o}_k(\boldsymbol{y}(\tau_k)) + \boldsymbol{c}_k + \boldsymbol{q}_k(\boldsymbol{x}_T(\tau_k) + \boldsymbol{y}(\tau_k), \xi) \\
& + \boldsymbol{v}_k(\boldsymbol{x}_T(\tau_k) + \boldsymbol{y}(\tau_k))]. \qquad (9.149)
\end{aligned}$$

It is easy to see that the operator $\mathcal{O}(\boldsymbol{x})$ is a contraction and therefore has a unique fixed point $\boldsymbol{y}_1(t) \in \mho(\varepsilon_0)$ which implies the estimate (9.145). From the fact that $\boldsymbol{y}_1(t)$ is a solution of system (9.147) we know $\boldsymbol{x}_T^{\xi}(t) = \boldsymbol{x}_T(t) + \boldsymbol{y}_1(t)$ is a T-periodic solution of system (9.141). Using the similar process as that used in the proof of Theorem 9.2.1 we can get the conclusion in (9.146). ■

9.2.3 Control Impulses at Variable Time

In this section let us study impulsive control of periodic motions when control impulses are at variable time. Let us consider the following T-periodic impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \xi), & t \neq \tau_k(\boldsymbol{x}, \xi), \\ \Delta\boldsymbol{x} = U(k, \boldsymbol{x}, \xi), & t = \tau_k(\boldsymbol{x}, \xi), \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}, \end{cases} \qquad (9.150)}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\xi \in \daleth \triangleq (-\tilde{\xi}, \tilde{\xi})$ is a small parameter perturbation that caused by parameter drift. We assume that there is a $T > 0$ and ϱ such that

$$\tau_k(\boldsymbol{x}, \xi) < \tau_{k+1}(\boldsymbol{x}, \xi), \quad \tau_{k+\varrho}(\boldsymbol{x}, \xi) = \tau_k(\boldsymbol{x}, \xi) + T$$

for all $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathbb{R}^n$ and $\xi \in \daleth$. $\boldsymbol{f} \in C^1[(\tau_k, \tau_{k+1}] \times \mathbb{R}^n \times \daleth, \mathbb{R}^n]$ for all $k \in \mathbb{Z}$ and

$$\lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \boldsymbol{f}(t, \boldsymbol{y}, \xi) < \infty, \qquad \lim_{(t,\boldsymbol{y})\to(\tau_k^+,\boldsymbol{x})} \frac{\partial \boldsymbol{f}(t, \boldsymbol{y}, \xi)}{\partial \boldsymbol{x}} < \infty,$$

$\boldsymbol{f}(t + T, \boldsymbol{x}, \xi) = \boldsymbol{f}(t, \boldsymbol{x}, \xi)$ for $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathbb{R}^n$, $\xi \in \daleth$ and $t \in \mathbb{R}$. $U \in C^1[\mathbb{Z} \times \mathbb{R}^n \times \daleth, \mathbb{R}^n]$ and

$$U(k+\varrho, \boldsymbol{x}, \xi) = U(k, \boldsymbol{x}, \xi) \text{ for } k \in \mathbb{Z}, \boldsymbol{x} \in \mathbb{R}^n, \xi \in \daleth.$$

Let us assume that control system (9.150) has a T-periodic solution $\boldsymbol{x}_T(t)$ with impulse moment at $\{\tau_k^T\}_{k=1}^{\varrho}$ when $\xi = 0$ and for each $k = 1, \cdots, \varrho$, $\tau_k(\boldsymbol{x}, \xi)$ is differentiable in some neighborhood of the point $(\boldsymbol{x}, \xi) = (\boldsymbol{x}_T(\tau_k^T), 0)$ and

$$\frac{\partial \tau_k(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\tau_k^T, \boldsymbol{x}_T(\tau_k^T), 0) \neq 1. \tag{9.151}$$

We then have the following reference system with respect to $\boldsymbol{x}_T(t)$:

$$\begin{aligned}
\dot{\boldsymbol{y}} &= \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}_T(t), 0)}{\partial \boldsymbol{x}} y, \quad t \neq \tau_k^T, \\
\Delta \boldsymbol{y} &= \Bigg[\frac{\partial U(k, \boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \\
&\quad + \Bigg(\frac{\partial U(k, \boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\tau_k^T, \boldsymbol{x}_T(\tau_k^T), 0) + \boldsymbol{f}(\tau_k^T, \boldsymbol{x}_T(\tau_k^T), 0) \\
&\quad - \boldsymbol{f}(\tau_k^{T+}, \boldsymbol{x}_T(\tau_k^{T+}), 0) \Bigg) \frac{\frac{\partial \tau_k(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}}}{1 - \frac{\partial \tau_k(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\tau_k^T, \boldsymbol{x}_T(\tau_k^T), 0)} \Bigg] \boldsymbol{y}, \\
&\quad t = \tau_k^T.
\end{aligned} \tag{9.152}$$

Theorem 9.2.4. *Let $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ be the solution of system (9.150) satisfying $\boldsymbol{x}(0, \boldsymbol{x}_0, \xi) = \boldsymbol{x}_0$ and assume that*

1. *The reference system (9.152) has no nontrivial T-periodic solution;*
2. *There is a $\varepsilon > 0$ such that for any $\xi \in (-\varepsilon, \varepsilon)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\|\boldsymbol{x}_0 - \boldsymbol{x}_T(0)\| < \varepsilon$, the solution $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ of (9.150) is defined on $t \in [0, T]$;*
3. *In some neighborhood of $(t, \boldsymbol{x}, \xi) = (T, \boldsymbol{x}_T(0), 0)$, $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ is continuous with respect to $(t, \boldsymbol{x}_0, \xi)$ and differentiable with respect to $\boldsymbol{x}_0$.*

Then, there is a $\xi_0 \in (0, \tilde{\xi})$ such that for $\xi \in [-\xi_0, \xi_0]$ system (9.150) has a unique T-periodic solution $\boldsymbol{x}_T^{\xi}(t)$. ⊠

Proof. Without loss of generality let us assume that $\tau_0^T(\boldsymbol{x}_T(0), 0) < 0 < \tau_1^T(\boldsymbol{x}_T(0), 0)$, then since $\boldsymbol{x}_T(t)$ is T-periodic, we have

$$\tau_{\varrho}^T(\boldsymbol{x}_T(T), 0) < T < \tau_{\varrho+1}^T(\boldsymbol{x}_T(T), 0).$$

It follows from assumption 3 that there is a $\varepsilon_1 \in (0, \varepsilon)$ such that for $|\boldsymbol{x}_0 - \boldsymbol{x}_T(0)| < \varepsilon_1$ and $|\xi| < \varepsilon_1$ we have

$$\tau_0^T(\boldsymbol{x}_0, \xi) < 0 < \tau_1^T(\boldsymbol{x}_0, \xi), \quad \tau_{\varrho}^T(\boldsymbol{x}(T, \boldsymbol{x}_0, \xi), \xi) < 0 < \tau_1^T(\boldsymbol{x}(T, \boldsymbol{x}_0, \xi), \xi).$$

Let us define the function

$$\boldsymbol{z}(\boldsymbol{x}_0, \xi) \triangleq \boldsymbol{x}_0 - \boldsymbol{x}(T, \boldsymbol{x}_0, \xi)$$

on set

$$\Omega \triangleq \{(\boldsymbol{x}_0, \xi) \in \mathbb{R}^n \times \daleth \mid \|\boldsymbol{x}_0 - \boldsymbol{x}_T(0)\| < \varepsilon_1, |\xi| < \varepsilon_1\}.$$

$\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ is T-periodic if and only if $\boldsymbol{x}(T, \boldsymbol{x}_0, \xi) = \boldsymbol{x}_0$; namely,

$$\boldsymbol{z}(\boldsymbol{x}_0, \xi) = 0. \tag{9.153}$$

It is clear that $\boldsymbol{z}(\boldsymbol{x}_T(0), \xi) = 0$ and it follows from assumption 3 that $\boldsymbol{z}(\boldsymbol{x}_0, \xi)$ is continuous with respect to $(\boldsymbol{x}_0, \xi)$ and differentiable with respect to $\boldsymbol{x}_0$ for every $(\boldsymbol{x}_0, \xi) \in \Omega$ and

$$\frac{\partial \boldsymbol{z}(\boldsymbol{x}_T(0), 0)}{\partial \boldsymbol{x}_0} = I - \frac{\partial \boldsymbol{x}(T, \boldsymbol{x}_T(0), 0)}{\partial \boldsymbol{x}_0}. \tag{9.154}$$

Since $\boldsymbol{y}(t) = \frac{\partial \boldsymbol{x}(t, \boldsymbol{x}_T(0), 0)}{\partial \boldsymbol{x}_0}$ satisfies (9.152) with initial condition $\boldsymbol{y}(0) = I$, we know that

$$\boldsymbol{y}(T) = \frac{\partial \boldsymbol{x}(T, \boldsymbol{x}_T(0), 0)}{\partial \boldsymbol{x}_0}$$

is a monodromy matrix of system (9.152). From assumption 1 we know that the following matrix is nonsingular:

$$I - \frac{\partial \boldsymbol{x}_T(\boldsymbol{x}_T(0), 0)}{\partial \boldsymbol{x}_0} = I - \boldsymbol{y}(T).$$

It then follows from implicit function theorem that there is a $\xi_0 \in (0, \tilde{\xi}) \cap (0, \varepsilon_1)$ such that for any $\xi \in (-\xi_0, \xi_0)$, (9.153) has a unique solution $\boldsymbol{x}_0 = \boldsymbol{x}_0(\xi)$ satisfying $\boldsymbol{x}_0(0) = \boldsymbol{x}_T(0)$. Thus, system (9.150) has a T-periodic solution

$$\boldsymbol{x}_T^{\xi}(t) = \boldsymbol{x}_T^{\xi}(t, \boldsymbol{x}_0(\xi), \xi).$$

■

Remark 9.2.1. Let the impulse moments of $\boldsymbol{x}_T^{\xi}(t)$ be $\{\tau_k^{\xi}\}$, then from the conditions of continuity in Theorem 9.2.4 we know that as $\xi \to 0$, $\boldsymbol{x}_T^{\xi}(t)$ approaches $\boldsymbol{x}_T(t)$ and τ_k^{ξ} approaches τ_k^T. ♦

Let us then consider the following impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \xi), & \Sigma(t, \boldsymbol{x}, \xi) \neq 0, \\ \Delta \boldsymbol{x} = U(t, \boldsymbol{x}, \xi), & \Sigma(t, \boldsymbol{x}, \xi) = 0, \quad t \in \mathbb{R}. \end{cases}} \tag{9.155}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\xi \in \daleth \triangleq (-\tilde{\xi}, \tilde{\xi})$ is a small parameter perturbation that caused by parameter drift and $\Sigma(t, \boldsymbol{x}, \xi) = 0$ defines a switching surface for each $\xi \in \daleth$. We assume that there is a $T > 0$ such that $\Sigma(t+T, \boldsymbol{x}, \xi) = \Sigma(t, \boldsymbol{x}, \xi)$. $\boldsymbol{f} \in C^1[\mathbb{R}_+ \times \mathbb{R}^n \times \daleth, \mathbb{R}^n]$, $\boldsymbol{f}(t+T, \boldsymbol{x}, \xi) = \boldsymbol{f}(t, \boldsymbol{x}, \xi)$ for $\boldsymbol{x} \in \mathbb{R}^n$, $\xi \in \daleth$ and $t \in \mathbb{R}$. $U \in C^1[\mathbb{R}_+ \times \mathbb{R}^n \times \daleth, \mathbb{R}^n]$ and

$$U(t+T,\boldsymbol{x},\xi)=U(t,\boldsymbol{x},\xi) \text{ for } t\in\mathbb{R},\boldsymbol{x}\in\mathbb{R}^n,\xi\in\beth.$$

Let us assume that control system (9.155) has a T-periodic solution $\boldsymbol{x}_T(t)$ with impulse moment at $\{\tau_k^T\}_{k=1}^{\varrho}$ when $\xi=0$ and there is a $\varrho\in\mathbb{N}$ such that

$$\tau_{k+\varrho}^T=\tau_k^T+T,\quad k\in\mathbb{N}.$$

We also assume that for each $k=1,\cdots,\varrho$, $\Sigma(t,\boldsymbol{x},\xi)$ is differentiable in some neighborhood of the point $(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)$ and

$$\frac{\partial\Sigma(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial\boldsymbol{x}}\boldsymbol{f}(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)+\frac{\partial\Sigma(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial t}\neq 0.$$

We then have the following reference system with respect to $\boldsymbol{x}_T(t)$:

$$\begin{aligned}\dot{\boldsymbol{y}}&=\frac{\partial\boldsymbol{f}(t,\boldsymbol{x}_T(t),0)}{\partial\boldsymbol{x}}y,\qquad t\neq\tau_k^T,\\ \Delta\boldsymbol{y}&=\left[\frac{\partial U(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial\boldsymbol{x}}\right.\\ &\quad+\left(-\frac{\partial U(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial\boldsymbol{x}}\boldsymbol{f}(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)-\boldsymbol{f}(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)\right.\\ &\quad\left.-\frac{\partial U(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial t}+\boldsymbol{f}(\tau_k^{T+},\boldsymbol{x}_T(\tau_k^{T+}),0)\right)\\ &\quad\left.\times\frac{\frac{\partial\Sigma(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial\boldsymbol{x}}}{\frac{\partial\Sigma(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial\boldsymbol{x}}\boldsymbol{f}(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)+\frac{\partial\Sigma(\tau_k^T,\boldsymbol{x}_T(\tau_k^T),0)}{\partial t}}\right]\boldsymbol{y},\\ &\quad t=\tau_k^T.\end{aligned}\tag{9.156}$$

Theorem 9.2.5. *Let $\boldsymbol{x}(t,\boldsymbol{x}_0,\xi)$ be the solution of system (9.155) satisfying $\boldsymbol{x}(0,\boldsymbol{x}_0,\xi)=\boldsymbol{x}_0$ and assume that*

1. *The reference system (9.156) has no non-trivial T-periodic solution;*
2. *There is a $\varepsilon>0$ such that for any $\xi\in(-\varepsilon,\varepsilon)$ and $\boldsymbol{x}_0\in\mathbb{R}^n$, $\|\boldsymbol{x}_0-\boldsymbol{x}_T(0)\|<\varepsilon$, the solution $\boldsymbol{x}(t,\boldsymbol{x}_0,\xi)$ of (9.155) is defined on $t\in[0,T]$;*
3. *In some neighborhood of $(t,\boldsymbol{x},\xi)=(T,\boldsymbol{x}_T(0),0)$, $\boldsymbol{x}(t,\boldsymbol{x}_0,\xi)$ is continuous with respect to $(t,\boldsymbol{x}_0,\xi)$ and differentiable with respect to $\boldsymbol{x}_0$;*
4. *$\Sigma(0,\boldsymbol{x}_T(0),0)\neq 0$.*

Then, there is a $\xi_0\in(0,\tilde{\xi})$ such that for $\xi\in[-\xi_0,\xi_0]$ system (9.155) has a unique T-periodic solution $\boldsymbol{x}_T^{\xi}(t)$. ⊠

Remark 9.2.2. The proof of Theorem 9.2.5 is similar to that of Theorem 9.2.4. Let the impulse moments of $\boldsymbol{x}_T^{\xi}(t)$ be $\{\tau_k^{\xi}\}$, then from the conditions of continuity in Theorem 9.2.5 we know that as $\xi\to 0$, $\boldsymbol{x}_T^{\xi}(t)$ approaches $\boldsymbol{x}_T(t)$ and τ_k^{ξ} approaches τ_k^T. ⧫

Let us then consider the following impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \xi), & \boldsymbol{x} \notin \Sigma(\xi), \\ \Delta \boldsymbol{x} = U(\boldsymbol{x}, \xi), & \boldsymbol{x} \in \Sigma(\xi) \end{cases}} \tag{9.157}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\xi \in \daleth \triangleq (-\tilde{\xi}, \tilde{\xi})$ is a small parameter perturbation that caused by parameter drift and for each $\xi \in \daleth$ the switching set $\Sigma(\xi)$ defines a hypersurface in $\mathbb{R}^n$. Let us assume that $\Sigma(\xi)$ consists of ϱ non-intersecting smooth hypersurfaces given by

$$\Sigma_k(\boldsymbol{x}, \xi) = 0, \quad k = 1, \cdots, \varrho.$$

$\boldsymbol{f} \in C^1[\mathbb{R}^n \times \daleth, \mathbb{R}^n]$ and $U \in C^1[\mathbb{R}^n \times \daleth, \mathbb{R}^n]$.

Let us assume that control system (9.157) has a T-periodic solution $\boldsymbol{x}_T(t)$ with impulse moment at $\{\tau_k^T\}_{k=1}^{\varrho}$ when $\xi = 0$ and

$$\tau_{k+\varrho}^T = \tau_k^T + T, \quad k \in \mathbb{Z}, \quad \dot{\boldsymbol{x}}_T(t) \neq 0 \text{ for some } t \in \mathbb{R}$$

and

$$\Sigma_k(\boldsymbol{x}_T(t), 0) = 0, \quad k = 1, \cdots, \varrho.$$

We also assume that for each $k = 1, \cdots, \varrho$, $\Sigma_k(\boldsymbol{x}, \xi)$ is differentiable in some neighborhood of the point $(\boldsymbol{x}_T(\tau_k^T), 0)$ and

$$\frac{\partial \Sigma(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^T), 0) \neq 0.$$

We then have the following reference system with respect to $\boldsymbol{x}_T(t)$:

$$\begin{aligned} \dot{\boldsymbol{y}} &= \frac{\partial \boldsymbol{f}(\boldsymbol{x}_T, 0)}{\partial \boldsymbol{x}} y, \quad t \neq \tau_k^T, \\ \Delta \boldsymbol{y} &= \left[\frac{\partial U(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \right. \\ &\quad + \left(-\frac{\partial U(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^T), 0) - \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^T), 0) \right. \\ &\quad \left. \left. + \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^{T+}), 0) \right) \times \frac{\frac{\partial \Sigma_k(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}}}{\frac{\partial \Sigma_k(\boldsymbol{x}_T(\tau_k^T), 0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^T), 0)} \right] \boldsymbol{y}, \\ &\quad t = \tau_k^T. \end{aligned} \tag{9.158}$$

We then show that $\dot{\boldsymbol{x}}_T(t)$ is a solution of the reference system (9.158). For $t \neq \tau_k^T$ we have

$$\frac{d\dot{\boldsymbol{x}}_T(t)}{dt} = \frac{\boldsymbol{f}(\boldsymbol{x}_T(t), 0)}{dt} = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_T, 0)}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}}_T(t) \tag{9.159}$$

which is the first equation of the reference system (9.158). For $t = \tau_k$ we have

$$\frac{\partial \Sigma_k(\boldsymbol{x}_T(\tau_k),0)}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}}_T(\tau_k) = \frac{\partial \Sigma_k(\boldsymbol{x}_T(\tau_k),0)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}_T(\tau_k),0), \tag{9.160}$$

and

$$\Delta \dot{\boldsymbol{x}}_T(\tau_k) = \dot{\boldsymbol{x}}_T(\tau_k^+) - \dot{\boldsymbol{x}}_T(\tau_k) = \boldsymbol{f}(\boldsymbol{x}_T(\tau_k^+),0) - \boldsymbol{f}(\boldsymbol{x}_T(\tau_k),0) \tag{9.161}$$

which leads to the second equation of the reference system (9.158).

Since $\dot{\boldsymbol{x}}_T(t) \neq 0$ for some $t \in \mathbb{R}$, it is a nontrivial T-periodic solution of system (9.158) and therefore system (9.158) has a multiplier equal to 1. We also assume that system (9.158) has no nontrivial T-periodic solution other than $\dot{\boldsymbol{x}}_T(t)$.

Theorem 9.2.6. *Let $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ be the solution of system (9.157) satisfying $\boldsymbol{x}(0, \boldsymbol{x}_0, \xi) = \boldsymbol{x}_0$ and assume that*

1. *There is a $\varepsilon > 0$ such that for any $\xi \in (-\varepsilon, \varepsilon)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\|\boldsymbol{x}_0 - \boldsymbol{x}_T(0)\| < \varepsilon$, the solution $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ of (9.157) is defined on $t \in [0, T+\xi]$;*
2. *In some neighborhood of $(t, \boldsymbol{x}, \xi) = (T, \boldsymbol{x}_T(0), 0)$, $\boldsymbol{x}(t, \boldsymbol{x}_0, \xi)$ is continuous with respect to $(t, \boldsymbol{x}_0, \xi)$ and differentiable with respect to $\boldsymbol{x}_0$.*

Then, there is a $\xi_0 \in (0, \tilde{\xi})$ such that for $\xi \in [-\xi_0, \xi_0]$ system (9.157) has a unique T^{ξ}-periodic solution $\boldsymbol{x}_T^{\xi}(t)$. ⊠

Proof. Without loss of generality let us suppose that $\dot{\boldsymbol{x}}_T(0) = \boldsymbol{e}_1 = (1, 0, \cdots, 0)^\top$ and $\boldsymbol{x}_T(0) = 0$[2]. Let $\boldsymbol{x}(t, \boldsymbol{x}_a, \xi)$ be a solution of system (9.157) satisfying

$$t \in [0, T+\varepsilon], \quad \xi \in (-\varepsilon, \varepsilon), \quad \boldsymbol{x}_a = (x_{a,1}, \cdots, x_{a,n})^\top, \quad \|\boldsymbol{x}_a\| < \varepsilon.$$

It follows from assumption 2 that

$$\frac{\partial \boldsymbol{x}(t,0,0)}{\partial x_{a,1}}$$

is a solution of the reference system (9.158) and

$$\frac{\partial \boldsymbol{x}(0,0,0)}{\partial x_{a,1}} = \boldsymbol{e}_1.$$

Since $\dot{\boldsymbol{x}}_T(t)$ is a solution of the reference system (9.158) with the initial condition $\dot{\boldsymbol{x}}_T(0) = \boldsymbol{e}_1$, we then have

$$\frac{\partial \boldsymbol{x}(t,0,0)}{\partial x_{a,1}} \equiv \dot{\boldsymbol{x}}_T(t) \text{ and } \frac{\partial \boldsymbol{x}(T,0,0)}{\partial x_{a,1}} = \dot{\boldsymbol{x}}_T(T) = \dot{\boldsymbol{x}}_T(0) = \boldsymbol{e}_1.$$

From the fact that

[2] This can be achieved by using linear transformations satisfying $\boldsymbol{x} = \Xi \boldsymbol{z} + \boldsymbol{x}_T(0)$, $\Xi \in \mathbb{R}^{n \times n}$, $\det \Xi \neq 0$ and $\Xi^{-1} \dot{\boldsymbol{x}}_T(0) = \boldsymbol{e}_1$.

$$\frac{\partial \boldsymbol{x}(T,0,0)}{\partial \boldsymbol{x}_a}$$

is a monodromy matrix of the reference system (9.158) we know that the multipliers, $\mu_i, i = 1, \cdots, n$, of the reference system (9.158) are roots of

$$\frac{\partial \boldsymbol{x}(T,0,0)}{\partial \boldsymbol{x}_a} - \mu I = 0;$$

namely,

$$\begin{aligned}
&\det\begin{pmatrix} 1-\mu & \frac{\partial x_1}{\partial x_{a,2}} & \cdots & \frac{\partial x_1}{\partial x_{a,n}} \\ 0 & \frac{\partial x_2}{\partial x_{a,2}} - \mu & \cdots & \frac{\partial x_2}{\partial x_{a,n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial x_n}{\partial x_{a,2}} & \cdots & \frac{\partial x_n}{\partial x_{a,n}} - \mu \end{pmatrix} \\
&= (1-\mu)\det\begin{pmatrix} \frac{\partial x_2}{\partial x_{a,2}} - \mu & \cdots & \frac{\partial x_2}{\partial x_{a,n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_{a,2}} & \cdots & \frac{\partial x_n}{\partial x_{a,n}} - \mu \end{pmatrix} \\
&\triangleq (1-\mu)D(\mu) = 0
\end{aligned} \tag{9.162}$$

where

$$\frac{\partial x_i}{\partial x_{a,j}} \triangleq \frac{\partial x_i(T,0,0)}{\partial x_{a,j}}. \tag{9.163}$$

Since $\mu = 1$ is a simple multiplier of the reference system (9.158) we know that $D(1) \neq 0$. Let us choose $\boldsymbol{x}_a$ from the hyperplane $x_{a,1} = 0$, then $\boldsymbol{x}(t, \boldsymbol{x}_a, \xi)$ is ϖ-periodic if and only if

$$\boldsymbol{w}(\varpi, \boldsymbol{x}_a, \xi) \triangleq \boldsymbol{x}(\varpi, \boldsymbol{x}_a, \xi) - \boldsymbol{x}_a = 0. \tag{9.164}$$

It is easy to see that $\boldsymbol{w}(T,0,0) = 0$. Because

$$\left.\frac{\partial \boldsymbol{x}(\varpi, \boldsymbol{x}_a, \xi)}{\partial \varpi}\right|_{\varpi=T} = \dot{\boldsymbol{x}}_T(T) = \boldsymbol{e}_1,$$

the Jacobian matrix of $\boldsymbol{w}$ with respect to variable $\boldsymbol{v} = (\varpi, x_{a,2}, \cdots, x_{a,n})^\top$ at point $(T,0,0)^\top$ is given by

$$\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{v}} = \begin{pmatrix} 1 & \frac{\partial x_1}{\partial x_{a,2}} & \cdots & \frac{\partial x_1}{\partial x_{a,n}} \\ 0 & \frac{\partial x_2}{\partial x_{a,2}} - 1 & \cdots & \frac{\partial x_2}{\partial x_{a,n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial x_n}{\partial x_{a,2}} & \cdots & \frac{\partial x_n}{\partial x_{a,n}} - 1 \end{pmatrix}$$

from which we have

$$\det \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{v}} = D(1) \le 0.$$

Thus, it follows from implicit function theorem that there is a $\xi_0 \in (0, \tilde{\xi})$ such that (9.164) has the following unique solution

$$(\varpi(\xi), x_{a,2}(\xi), \cdots, x_{a,n}(\xi))^\top$$

for $\xi \in [-\xi_0, \xi_0]$ and satisfying

$$\varpi(0) = T, x_{a,2}(0) = 0, \cdots, x_{a,n}(0) = 0.$$

Let us set $T^\xi = \varpi(\xi)$, then system (9.157) has a T^ξ-periodic solution with initial condition as $(0, x_{a,2}(\xi), \cdots, x_{a,n}(\xi))^\top$. ■

Remark 9.2.3. T^ξ approaches T as $\xi \to 0$. Let the impulse moments of $\boldsymbol{x}_T^\xi(t)$ be $\{\tau_k^\xi\}$, we know that as $\xi \to 0$, $\boldsymbol{x}_T^\xi(t)$ approaches $\boldsymbol{x}_T(t)$ and τ_k^ξ approaches τ_k^T. It is also clear that if $\xi = 0$ and the multipliers of system (9.158) satisfy

$$\mu_1 = 1, \quad |\mu_i| < 1, \quad i = 2, \cdots, n$$

then the T-periodic solution of system (9.157) with $\xi = 0$ is orbitally asymptotically stable. ♦

Let us consider the following periodic impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \ne \tau_k \\ \Delta \boldsymbol{x} &= U(k, \boldsymbol{x}), \quad t = \tau_k \end{aligned} \qquad (9.165)}$$

where $\boldsymbol{x} \in \Omega \subset \mathbb{R}^n$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. $\boldsymbol{f} \in C[(\tau_k, \tau_{k+1}] \times \Omega, \mathbb{R}^n]$ for $k \in \mathbb{Z}$ and for each $\boldsymbol{x} \in \Omega$ we have

$$\lim_{(t,\boldsymbol{y}) \to (\tau_k^+, \boldsymbol{x})} \|\boldsymbol{f}(t, \boldsymbol{y})\| < \infty, \quad k \in \mathbb{Z}. \tag{9.166}$$

There is a $T > 0$ such that

$$\boldsymbol{f}(t + T, \boldsymbol{x}) = \boldsymbol{f}(t, \boldsymbol{x}). \tag{9.167}$$

There is a ϱ such that $U \in C[\mathbb{Z} \times \Omega, \mathbb{R}^n]$ satisfies

$$U(k + \varrho, \boldsymbol{x}) = U(k, \boldsymbol{x}), \quad k \in \mathbb{Z}, \quad \boldsymbol{x} \in \Omega.$$

And let us assume that

$$\tau_{k+\varrho} = \tau_k + T, \quad \tau_0 < 0 < \tau_1 < \cdots < \tau_\varrho < \tau_{\varrho+1}.$$

For the sake of self-contained, we then present some useful theorems without proofs which can be found in [3].

Theorem 9.2.7. *Let the following conditions hold:*

1. $\boldsymbol{f}(t,\boldsymbol{x})$ *is locally Lipschitzian with respect to* $\boldsymbol{x}$ *in domain* $\mathbb{R}\times\Omega$*;*
2. *Let* S *be a domain in* $\mathbb{R}^n$ *with boundary* $\bigcirc$ *and closure* $\textcircled{S} = S\cup\bigcirc\subset\Omega$*.* S *is a bounded and convex set and* $\textcircled{S}$ *is defined by a finite number of inequalities*
$$\eth_i(\boldsymbol{x})\le 0,\quad i=1,\cdots,r,$$
where $\eth_i:\mathbb{R}^n\to\mathbb{R},\ i=1,\cdots,r,$ *are smooth. If for* $\boldsymbol{x}\in\bigcirc$ *we have*
$$\eth_i(\boldsymbol{x})=0,\ \textit{for some } i=1,\cdots,r,$$
then
$$\frac{\partial\eth_i(\boldsymbol{x})}{\partial\boldsymbol{x}}\neq 0.$$
For any $\boldsymbol{x}\in\bigcirc$ *let us define the set*
$$D(\boldsymbol{x})\triangleq\{i\in\{1,\cdots,r\}\mid\eth_i(\boldsymbol{x})=0\};$$
3. *For* $t\in\mathbb{R},\ \boldsymbol{x}\in\bigcirc$ *and* $i\in D(\boldsymbol{x})$ *we have*
$$\frac{\partial\eth_i(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(t,\boldsymbol{x})<0;$$
4. $\eth_i(\boldsymbol{x}+U(k,\boldsymbol{x})\le 0$ *for* $\boldsymbol{x}\in\textcircled{S}$*,* $i=1,\cdots,r,$ *and* $k=1,\cdots,\varrho$*.*

Then the control system (9.165) has a T*-periodic solution* $\boldsymbol{x}_T(t)\in\textcircled{S}$*.* ⊠

Theorem 9.2.8. *Let the following conditions hold:*

1. *Let* S *be a domain in* $\mathbb{R}^n$ *with boundary* $\bigcirc$ *and closure* $\textcircled{S} = S\cup\bigcirc\subset\Omega$*.* S *is a bounded and convex set and* $\textcircled{S}$ *is defined by a finite number of inequalities*
$$\eth_i(\boldsymbol{x})\le 0,\quad i=1,\cdots,r,$$
where $\eth_i:\mathbb{R}^n\to\mathbb{R}, i=1,\cdots,r,$ *are smooth. If for* $\boldsymbol{x}\in\bigcirc$ *we have*
$$\eth_i(\boldsymbol{x})=0,\ \textit{for some } i=1,\cdots,r,$$
then
$$\frac{\partial\eth_i(\boldsymbol{x})}{\partial\boldsymbol{x}}\neq 0.$$
For any $\boldsymbol{x}\in\bigcirc$ *let us define the set*
$$D(\boldsymbol{x})\triangleq\{i\in\{1,\cdots,r\}\mid\eth_i(\boldsymbol{x})=0\};$$
2. *For* $t\in\mathbb{R},\ \boldsymbol{x}\in\bigcirc$ *and* $i\in D(\boldsymbol{x})$ *we have*
$$\frac{\partial\eth_i(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(t,\boldsymbol{x})\le 0;$$

3. $\eth_i(\boldsymbol{x} + U(k, \boldsymbol{x}) \le 0$ *for* $\boldsymbol{x} \in \circledS$, $i = 1, \cdots, r$, *and* $k = 1, \cdots, \varrho$.

Then the control system (9.165) has a T-periodic solution $\boldsymbol{x}_T(t) \in \circledS$. ⊠

Theorem 9.2.9. *Let the following conditions hold:*

1. *Let* S *be a domain in* $\mathbb{R}^n$ *with boundary* $\bigcirc$ *and closure* $\circledS = S \cup \bigcirc \subset \Omega$. S *is a bounded and convex set and* $\circledS$ *is defined by a finite number of inequalities*
$$\eth_i(\boldsymbol{x}) \le 0, \quad i = 1, \cdots, r,$$
where $\eth_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \cdots, r$, *are smooth. If for* $\boldsymbol{x} \in \bigcirc$ *we have*
$$\eth_i(\boldsymbol{x}) = 0, \text{ for some } i = 1, \cdots, r,$$
then
$$\frac{\partial \eth_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \ne 0.$$
For any $\boldsymbol{x} \in \bigcirc$ *let us define the set*
$$D(\boldsymbol{x}) \triangleq \{i \in \{1, \cdots, r\} \mid \eth_i(\boldsymbol{x}) = 0\};$$
2. *For* $t \in \mathbb{R}$, $\boldsymbol{x} \in \bigcirc$ *and* $i \in D(\boldsymbol{x})$ *we have*
$$\frac{\partial \eth_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(t, \boldsymbol{x}) \ge 0;$$
3. *Let* $\Psi_k(\boldsymbol{x}) = I + U(k, \boldsymbol{x})$ *and* $\Psi_k : \circledS \to \mathbb{R}^n$, $k = 1, \cdots, \varrho$, *are homeomorphisms and* $\circledS \subset \Psi_k(\circledS)$ *for* $k = 1, \cdots, \varrho$.

Then the control system (9.165) has a T-periodic solution $\boldsymbol{x}_T(t) \in \circledS$. ⊠

Let us consider the following periodic impulsive control system:

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \quad t \ne \tau_k, \\ \boldsymbol{x}(\tau_k^+) &= \Psi_k(\boldsymbol{x}(\tau_k)), \quad t = \tau_k \end{aligned} \qquad (9.168)}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let there be a $T > 0$ and a $\varrho \in \mathbb{N}$ and let us denote $\daleth_0 \triangleq [0, \tau_1]$, $\daleth_i \triangleq (\tau_i, \tau_{i+1}], i = 1, \cdots, \varrho - 1$, and $\daleth_\varrho \triangleq (\tau_\varrho, T]$, $\daleth \triangleq [0, T]$, then we have the following theorem.

Theorem 9.2.10. *Assume that*

1. $\boldsymbol{v}(t)$ *and* $\boldsymbol{w}(t)$ *are the lower and upper solutions of system (9.168), respectively, such that* $\boldsymbol{v}(t) \preceq \boldsymbol{w}(t)$ *for* $t \in \daleth$;
2. $\boldsymbol{f} : \daleth \times \mathbb{R}^n \to \mathbb{R}^n$ *is quasimonotone nondecreasing in* $\daleth \times \mathbb{R}^n$, *and is continuous in* $\daleth_i \times \mathbb{R}^n$ *for* $i = 0, \cdots, \varrho$, *and for each* $k = 1, \cdots, \varrho$ *and* $\boldsymbol{x} \in \mathbb{R}^n$
$$\lim_{(t,\boldsymbol{y}) \to (\tau_k^+), \boldsymbol{x}} \|\boldsymbol{f}(t, \boldsymbol{y})\| < \infty;$$

3. There is a $\gamma \in \mathcal{L}^1[\beth, \mathbb{R}_+]$ *such that*

$$\sup_{\boldsymbol{v}(t) \preceq \boldsymbol{x}(t) \preceq \boldsymbol{w}(t)} |f_i(t, \boldsymbol{x})| \leq \gamma(t), \quad i = 1, \cdots, n$$

holds almost everywhere on $\beth$*;*

4. $\Psi_k : \mathbb{R}^n \to \mathbb{R}^n, k = 1, \cdots, \varrho$*, are continuous and nonincreasing with respect to* $\boldsymbol{x}$*.*

Then system (9.168) has a T*-periodic solution* $\boldsymbol{x}_T(t)$ *for* $t \in \beth$ *such that* $\boldsymbol{v}(t) \preceq \boldsymbol{x}_T(t) \preceq \boldsymbol{w}(t)$*.* ⊠

9.3 Applications

In this section we will present some applications of periodic impulsive control system to the control of chaotic systems to periodic motions.

9.3.1 Control Rössler System to Periodic Motions

In this section, we present the theoretical result of impulsive control of the Rössler system to periodic motions. A sufficient condition for existence of periodic trajectory of impulsively controlled Rössler system is given. Based on the theoretical results, we present a systematic method of designing impulsive control laws. We present the proportional and additive impulsive control laws as two examples to show how an impulsive control law can be designed.

The Rössler system[26] is given by

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = zx + b - cz \end{cases} \tag{9.169}$$

where a, b, and c are three parameters.

The impulsively controlled Rössler system is given by

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = zx + b - cz \end{cases}, \quad t \neq \tau_k, \quad t \in J, \tag{9.170}$$

and

$$\boldsymbol{x}(\tau_k^+) = \Psi_k(\boldsymbol{x}(\tau_k)), \quad k = 1, \cdots, q, \tag{9.171}$$

where Ψ_k is a continuous and nondecreasing function. Then we use the following theorem to guarantee that there exist periodic motions in the impulsively controlled system.

Theorem 9.3.1. *Assume that $a > 0$, $b > 0$, $c > 0$ and the free Rössler system in (9.169) is chaotic and $z > 0$, then the Rössler system can be impulsively suppressed into periodic trajectories.* ⊠

Proof. Let $(x, y, z)^\top$ denote the solution of the free Rössler system, since $z > 0$, we choose the lower solution $\boldsymbol{v}$ as

$$\begin{cases} v_1 = x \\ v_2 = y \\ v_3 = 0 \end{cases} \tag{9.172}$$

and choose the upper solution $\boldsymbol{w}$ as

$$\begin{cases} w_1 = x \\ w_2 = y \\ w_3 = 2z \end{cases} \tag{9.173}$$

then condition 1 of Theorem 9.2.10 is satisfied.

Let $\boldsymbol{u}^\top = (x_1, y, z)$, $\boldsymbol{v}^\top = (x_2, y, z)$, $x_1 < x_2$, and $f_2(t, \boldsymbol{x}) = x + ay$, $f_3(t, \boldsymbol{x}) = b + z(x - c)$ then

$$\begin{aligned} f_2(t, \boldsymbol{u}) - f_2(t, \boldsymbol{v}) &= (x_1 - x_2) < 0, \\ f_3(t, \boldsymbol{u}) - f_3(t, \boldsymbol{v}) &= [b + z(x_1 - c)] - [b + z(x_2 - c)] \\ &= z(x_1 - x_2) < 0. \end{aligned} \tag{9.174}$$

The last inequality is because of $z > 0$. Thus condition 2 of Theorem 9.2.10 is satisfied.

Since the free Rössler system is chaotic, all its state variables are bounded, condition 3 of Theorem 9.2.10 is satisfied.

Since the impulse generating law Ψ_k is a continuous and nondecreasing function, condition 4 of Theorem 9.2.10 is satisfied.

Then from Theorem 9.2.10, we know that the existence of periodic trajectories of the controlled Rössler system is guaranteed. ■

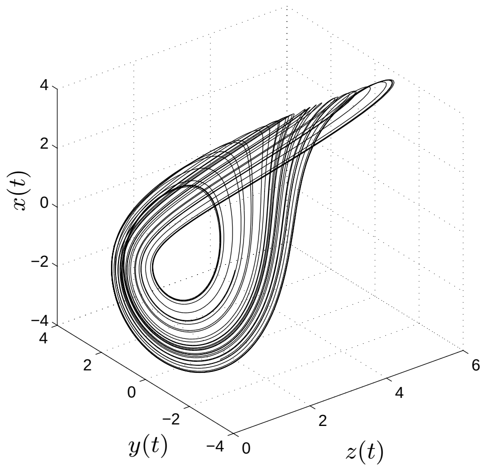
Numerical Experiments

The numerical experiments are as follows. In these experiments, we choose the parameters as $a = 0.398$, $b = 2$, and $c = 4.0$. The fourth-order Runge-Kutta algorithm with step size of 0.005 is used. The initial condition is given by $(x(0), y(0), z(0)) = (-2.277838, -2.696438, 0.304911)$.

Figure 9.4 shows the results when the following proportional impulse generating law at a Poincaré section is used. The Poincaré section is chosen as $x = 0$, and the proportional impulses are generated by

$$\begin{cases} x(\tau_k^+) = x(\tau_k) \\ y(\tau_k^+) = \psi_2(\boldsymbol{x}(\tau_k)) = [1 - \lambda \text{sgn}(y(\tau_k))] y(\tau_k), \quad |\lambda| < 1 \\ z(\tau_k^+) = z(\tau_k) \end{cases} \tag{9.175}$$

which is continuous and nondecreasing. Here sgn($\cdot$) denotes the signum function. Figure 9.4(a) shows the Rössler attractor of the free system. One can see that $z > 0$ is satisfied. Figure 9.4(b) shows the bifurcation diagram with respect to the control parameter λ, from which one can find the period 1, 2, and 4 windows very easily. Figure 9.4(c) shows the controlled period 1 trajectory when $\lambda = 0.65$. Figure 9.4(d) shows the controlled period 2 trajectory when $\lambda = 0.45$. Figure 9.4(e) shows the controlled period 4 trajectory when $\lambda = 0.35$. Figure 9.4(f) shows the controlled period 8 trajectory when $\lambda = 0.305$. We also found that if λ is much greater than 0.77 then the controlled Rössler system becomes unstable.



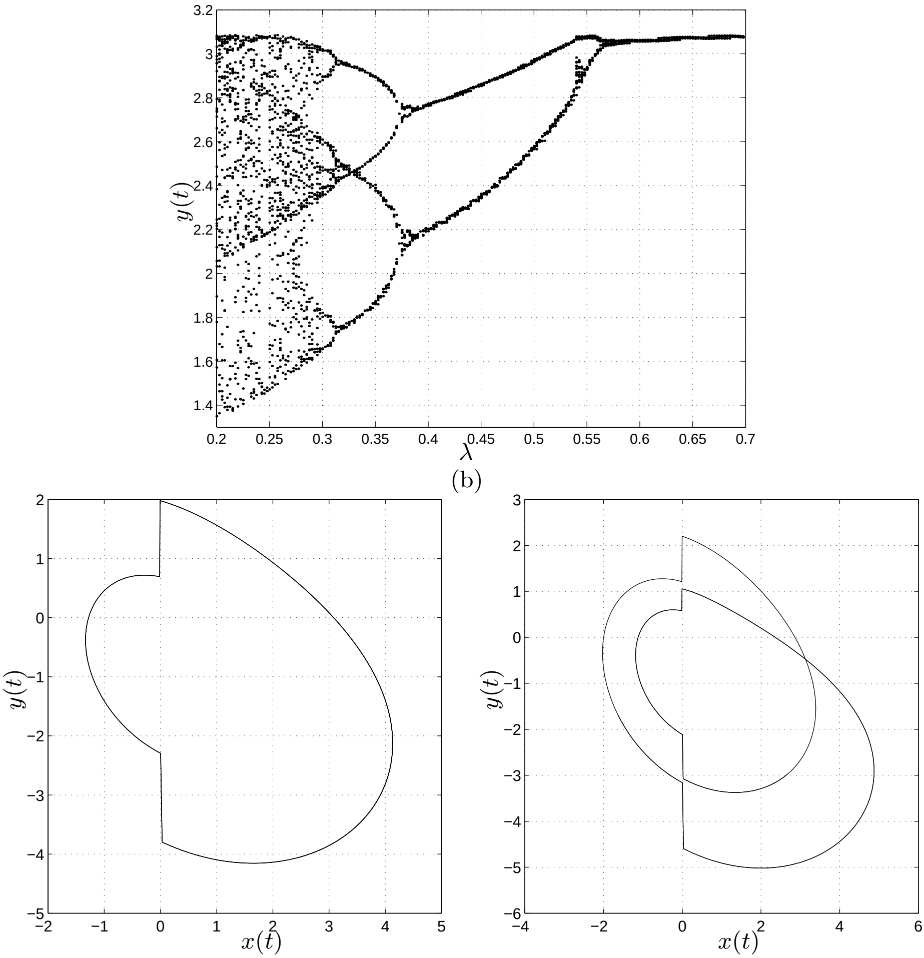
(a)

Fig. 9.4. Impulsive control of the Rössler system to periodic trajectories using proportional impulses at Poincaré section $x = 0$. (a) Attractor of the free Rössler system. (b) The bifurcation diagram of the impulsively controlled Rössler system. (c) The controlled period 1 trajectory. (d) The controlled period 2 trajectory. (e) The controlled period 4 trajectory. (f) The controlled period 8 trajectory.

We can also choose some other continuous and nondecreasing impulse generating laws. One of them is given by

$$\begin{cases} x(\tau_k^+) = x(\tau_k), \\ y(\tau_k^+) = \lambda + y(\tau_k), \\ z(\tau_k^+) = z(\tau_k), \end{cases} \tag{9.176}$$

which is called *additive impulse generating law*. Figure 9.5(a) shows the bifurcation diagram with respect to the control parameter λ. From Fig. 9.5(a) one can find the period 3 and period 6 windows. Figure 9.5(b) shows the controlled period 3 trajectory when $\lambda = 0.303$. Figure 9.5(c) shows the controlled period 6 trajectory when $\lambda = 0.45$.



(b)

(c) (d)

Fig. 9.4 (*Continued*)

Remark 9.3.1. The results in this section are adopted from [51]. In [11] the same method was used to control the following chaotic system[57] into periodic motions:

$$\begin{cases} \dot{x} = -x + \epsilon f(x) - \zeta f(y) + \eta \sin(\omega t) \\ \dot{y} = -y + \epsilon f(y) + \zeta f(x) \end{cases} \tag{9.177}$$

where $f(\cdot)$ is a piecewise linear function given by

$$f(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{9.178}$$

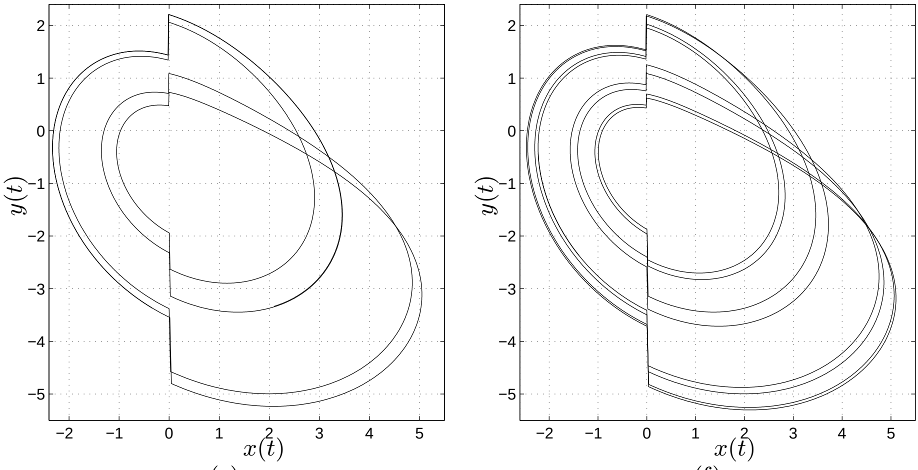


Fig. 9.4 (*Continued*)

We choose the parameters as: $\epsilon = 2$, $\zeta = 1.2$, $\omega = \pi/2$, and $\eta = 4.04$. The uncontrolled system is chaotic with a chaotic attractor shown in Fig.9.6.

Example 9.3.1. In this example, the impulsively controlled chaotic system is given by

$$\left.\begin{array}{l}\dot{x} = -x + \epsilon f(x) - \zeta f(y) + \eta \sin(\omega t)\\ \dot{y} = -y + \epsilon f(y) + \zeta f(x)\end{array}\right\}, \quad t \neq \tau_i,$$
$$\left.\begin{array}{l}x(\tau_i^+) = dx(\tau_i)\\ y(\tau_i^+) = dy(\tau_i)\end{array}\right\}, \quad d > 0, \quad t = \tau_i,$$
$$\text{switching set } \varSigma_i \text{ given by } x(\tau_i) = 0. \tag{9.179}$$

This kind of impulsive control law is called *proportional impulsive control law*. The simulation results with $d = 0.2$ are shown in Fig. 9.7. Observe that the controlled system approaches a period-1 trajectory. ★

Example 9.3.2. In this example, the impulsively controlled chaotic system is given by

$$\left.\begin{array}{l}\dot{x} = -x + \epsilon f(x) - \zeta f(y) + \eta \sin(\omega t)\\ \dot{y} = -y + \epsilon f(y) + \zeta f(x)\end{array}\right\}, \quad t \neq \tau_i,$$
$$\left.\begin{array}{l}x(\tau_i^+) = d + x(\tau_i)\\ y(\tau_i^+) = d + y(\tau_i)\end{array}\right\}, \quad d > 0, \quad t = \tau_i,$$
$$\text{switching set } \varSigma_i \text{ given by } x(\tau_i) = 0. \tag{9.180}$$

This kind of impulsive control law is called *additive impulsive control law*. The simulation results with $d = 0.6$ are shown in Fig. 9.8. Observe that the controlled system approaches a period-1 trajectory. ★ ♦

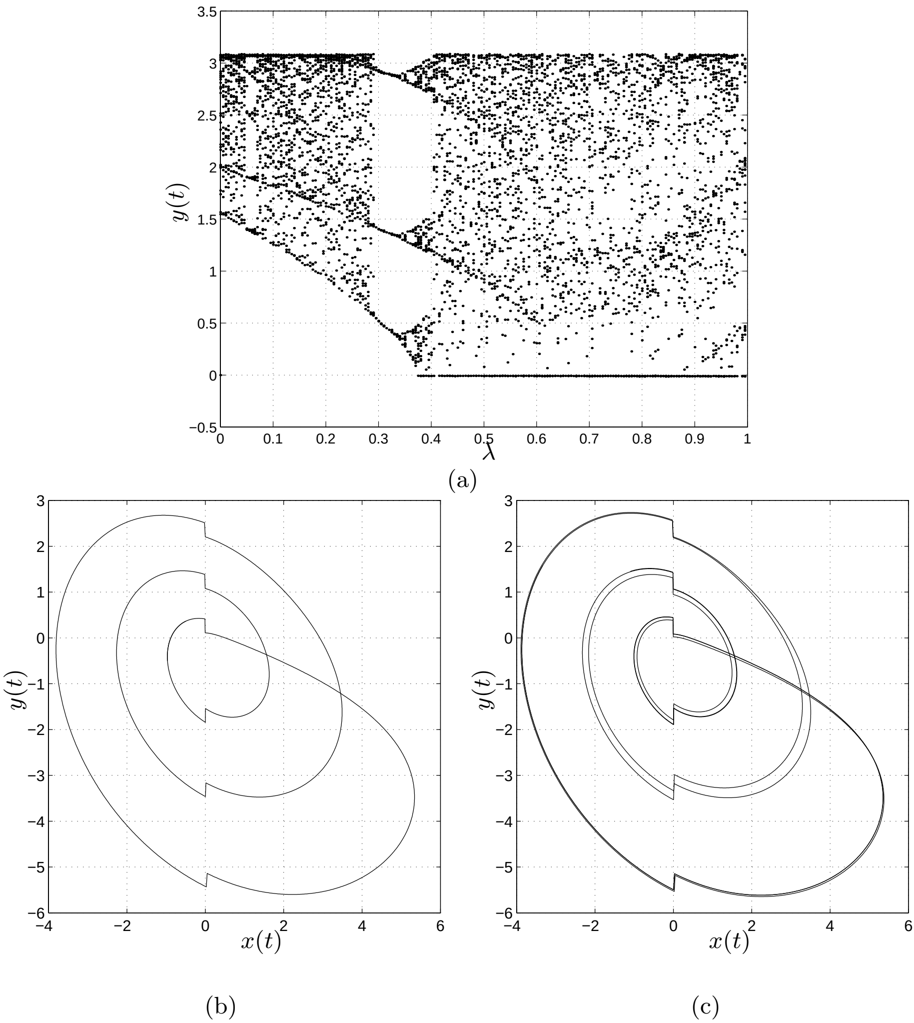


Fig. 9.5. Impulsive control of the Rössler system to periodic trajectories using additive impulses at Poincaré section $x = 0$. (a) The bifurcation diagram of the impulsively controlled Rössler system. (b) The controlled period 3 trajectory. (c) The controlled period 6 trajectory.

9.3.2 Control of Stepping Motor

A stepping motor can be modeled as a periodic impulsive control system with periodic control impulses. Let us consider the following model of a two-phase hybrid stepping motor:

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + T_L = \sqrt{2}KI_m \sin(N(U(t) - \theta)) \tag{9.181}$$

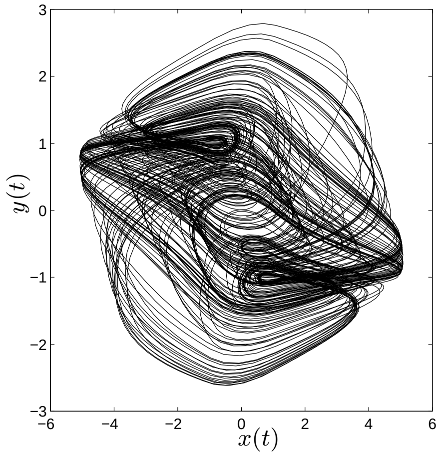


Fig. 9.6. The chaotic attractor of the uncontrolled chaotic system in (9.177).

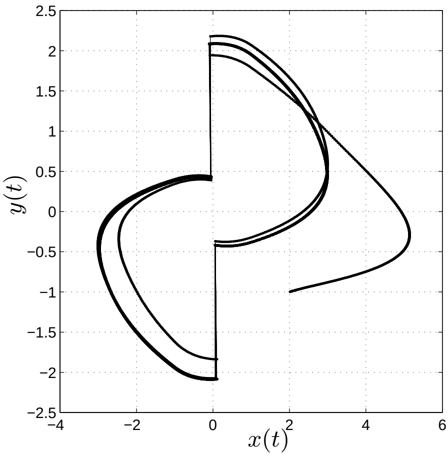


Fig. 9.7. Impulsive stabilization of period-1 trajectories with $d = 0.2$.

where

J	[kg· cm· s^2]	the moment of inertia,
D	[kg· cm/rad/s]	viscous friction coefficient,
T_L	[kg· cm]	load torque,
K	[kg· cm/A]	torque constant,
I_m	[A]	maximum motor current,
N		number of rotor teeth,
U	[rad]	mechanical angle,

and

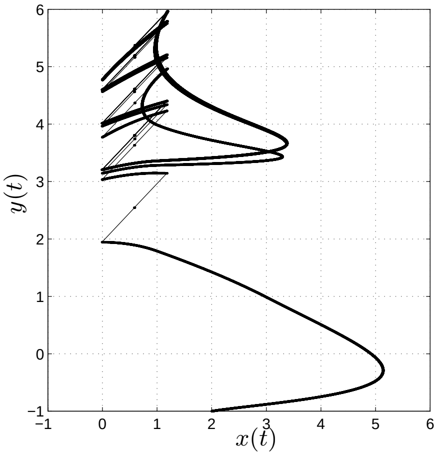


Fig. 9.8. Impulsive stabilization of period-1 trajectories with $d = 0.6$.

$$U(t) = \frac{\pi h}{2\pi} \sum_{k=0}^{\infty} u_{\text{step}}(t - kT) \tag{9.182}$$

where $u_{\text{step}}(t)$ is the unit step function. With $\phi = N\theta$ we can rewrite (9.181) as

$$\frac{d^2\phi}{dt^2} + \frac{D}{J}\frac{d\phi}{dt}\frac{N\sqrt{2}KI_m}{J}\sin(\phi - NU(t)) + \frac{NT_L}{J} = 0 \tag{9.183}$$

Let

$$\omega = \sqrt{\frac{N\sqrt{2}KI_m}{J}}$$

and $s = \omega t$, (9.183) can be rewritten as

$$\frac{d^2\phi}{ds^2} + \kappa\frac{d\phi}{dt} + \sin\left(\phi - NU\left(\frac{s}{\omega}\right)\right) + B_0 = 0 \tag{9.184}$$

where

$$\kappa = \frac{D}{\sqrt{N\sqrt{2}KI_m J}}, \qquad B_0 = \frac{T_L}{\sqrt{2}KI_m}. \tag{9.185}$$

Let us choose variables in (9.184) as

$$t = s, \quad x = \phi - NU\left(\frac{s}{\omega}\right), \quad y = \frac{d\phi}{ds} \tag{9.186}$$

then (9.183) can be written as

$$\left.\begin{array}{l}\dot{x}=y\\ \dot{y}=-\kappa y-\sin(x)-B_0\end{array}\right\},\quad t\neq k\omega T,$$
$$\Delta x=u_k,\quad t=k\omega T,\quad k\in\mathbb{N},$$
$$x(0)=y(0)=0 \tag{9.187}$$

where $x \in S^1 \triangleq \{x \in \mathbb{R} \bmod 2\pi\}$ and $y \in \mathbb{R}$, $u_k = -\frac{\pi}{2}$ is assumed. This stepping motor model can have very complex behaviors such as chaotic attractors as shown in Fig. 9.9.

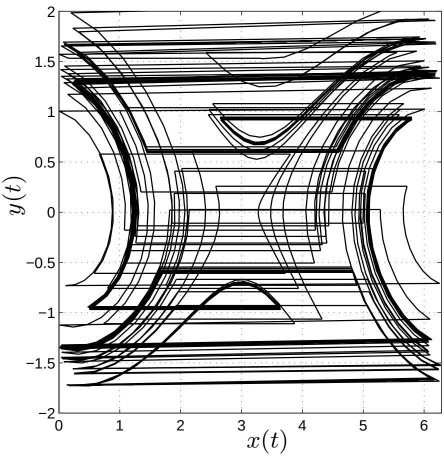


Fig. 9.9. Chaotic attractor of the stepping motor model. Parameters are $\kappa = 0.2$, $u_k = \pi$, $B_0 = 0$ and $\omega T = 4.5$.

However, since a stepping motor should be operated along periodic solutions, we need to control it into a periodic trajectory if it falls into chaotic region. This problem is called stabilizing unstable periodic trajectories embedded in a chaotic attractor. Since a periodic solution corresponding to a fixed point of the Poincaré map of the stepping motor model, we need to study the stability of fixed points of the following Poincaré map:

$$\Pi : \boldsymbol{x}(k\omega T) \mapsto \boldsymbol{x}((k+1)\omega T). \tag{9.188}$$

Let us denote $\boldsymbol{x}(k\omega T)$ as $\boldsymbol{x}_k$, then we can simplify (9.188) as

$$\Pi : \boldsymbol{x}_k \mapsto \boldsymbol{x}_{k+1}. \tag{9.189}$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix}$$

then from (9.187) we have

$$\boldsymbol{x}_{k+1} = e^{A\omega T}(\boldsymbol{x}_k + (u_k,\ 0)^\top) + \int_{k\omega T}^{(k+1)\omega T} e^{A((k+1)\omega T - s)} \begin{pmatrix} 0 \\ -\sin(x(s)) - B_0 \end{pmatrix} ds. \tag{9.190}$$

Assume $u_k = u$ is a constant, then without loss of generality we have

$$\boldsymbol{x}_{k+1} = e^{A\omega T}(\boldsymbol{x}_k + (u,\ 0)^\top) + \int_0^{\omega T} e^{A(\omega T - s)} \begin{pmatrix} 0 \\ -\sin(x(s)) - B_0 \end{pmatrix} ds. \tag{9.191}$$

where $\boldsymbol{x}(0^+) = \boldsymbol{x}_k + (u,\ 0)^\top$ is the initial condition for the integration.

We then control a chaotic trajectory to a target periodic trajectory using parameter perturbation method. For this purpose, let us rewrite the Pointcaré map as

$$\boldsymbol{x}_{k+1} = \Pi(\boldsymbol{x}_k, \boldsymbol{p})$$

where $\boldsymbol{p} \in \mathbb{R}^m$ is the parameter vector. Let us suppose that there is a target periodic trajectory as a fixed point $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfying

$$\boldsymbol{x}^* = \Pi(\boldsymbol{x}^*, \boldsymbol{p}^*),$$

then the variational equation in the neighborhood of $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is given by

$$\boldsymbol{x}_k = \boldsymbol{x}^* + \delta_k, \qquad \boldsymbol{p} = \boldsymbol{p}^* + \boldsymbol{p}_k.$$

We then have

$$\begin{aligned} \boldsymbol{x}^* + \delta_{k+1} &= \Pi(\boldsymbol{x}^* + \delta_k, \boldsymbol{p}^* + \boldsymbol{p}_k) \\ &= \Pi(\boldsymbol{x}^*, \boldsymbol{p}^*) + A_\Pi \delta_k + B_\Pi \boldsymbol{p}_k + \cdots \end{aligned} \tag{9.192}$$

where

$$A_\Pi = \left.\frac{\partial \Pi}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}^*, \boldsymbol{p}=\boldsymbol{p}^*}, \qquad B_\Pi = \left.\frac{\partial \Pi}{\partial \boldsymbol{p}}\right|_{\boldsymbol{x}=\boldsymbol{x}^*, \boldsymbol{p}=\boldsymbol{p}^*}. \tag{9.193}$$

Therefore, we have the following linearized difference equation in the neighborhood of the target periodic trajectory as

$$\delta_{k+1} = A_\Pi \delta_k + B_\Pi \boldsymbol{p}_k. \tag{9.194}$$

Assume that (A_Π, B_Π) is controllable, then we can design the following linear control law

$$\boldsymbol{p}_k = C\delta_k \tag{9.195}$$

such that $(A_\Pi + B_\Pi C)$ is stable. One design example based on this method can be found in [29].

Remark 9.3.2. In [10] the authors studied the stability of the trivial solution of system (9.187) under assumption of $B_0 = 0$ by using type-I impulsive control law. Let us rewrite system (9.187) as

$$\boxed{\begin{aligned} &\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\kappa y - \sin(x)\end{aligned}\right\}, \quad t \neq k\omega T,\\ &\Delta x = u_k, \quad t = k\omega T, \quad k \in \mathbb{N},\\ &\boldsymbol{x}_0 = (x(0), y(0))^\top \end{aligned} \qquad (9.196)}$$

where we use notation $\boldsymbol{x} = (x, y)^\top$ and let $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ be any a solution of impulsive control system (9.196).

Definition 9.3.1. *The impulsive control system (9.196) is said to be eventually exponentially asymptotically stable, if for each $\varepsilon > 0$, there exist three positive numbers, $\gamma, \delta = \delta(\varepsilon)$ and $T = T(\varepsilon)$, such that $\|\boldsymbol{x}_0\| < \delta$ implies*

$$\|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| < \varepsilon e^{-\gamma(t-t_0)}$$

for $t \geq t_0 \geq T$. ⊠

We then have the following conclusions whose proof can be found in [10].

Theorem 9.3.2. *If there exist two constants, μ and β, satisfying*

$$0 < \mu < \frac{2b}{3} + b^2 \text{ and } 1 < \beta < e^{\mu\omega T},$$

such that

$$\sum_{k=1}^{\infty} \frac{u_k^2 e^{k\mu\omega T}}{\beta^k} < \infty$$

then

1. $$\lim_{t\to\infty} \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)\| = 0;$$
2. *the impulsive control system (9.196) is eventually exponentially asymptotically stable.*

⊠

Corollary 9.3.1. *If there exists a constant $q > 0$ such that*

$$\left|\frac{u_{k+1}}{u_k}\right| \leq q < 1$$

for $k \geq N$, $k \in \mathbb{N}$, where N is a sufficiently large integer, then the conclusions of Theorem 9.3.2 hold. ⊠

One simulation result can be found in [10].

♦

10. Impulsive Control of Almost Periodic Motions

In this chapter we study impulsive control of almost periodic systems. The results can be used to control chaotic systems and design nanoelectronic circuits.

10.1 Almost Periodic Sequences and Functions

In this section we present some basic knowledge of almost periodic sequences and functions. To save space, only the immediately useful results will be presented.

Definition 10.1.1. *Let $\epsilon > 0$ and $\{\boldsymbol{x}_i\}, i \in \mathbb{Z}$ be a sequence in $\mathbb{R}^n$, an integer ϱ is called an ϵ-almost period of the sequence $\{\boldsymbol{x}_i\}$ if for each $k \in \mathbb{Z}$ we have*

$$\|\boldsymbol{x}_{k+\varrho} - \boldsymbol{x}_k\| < \epsilon. \tag{10.1}$$

$\boxtimes$

Definition 10.1.2. *A sequence $\{\boldsymbol{x}_i\}$ is almost periodic if for any $\epsilon > 0$ there is a relatively dense set of its ϵ-periods; namely, there is such a natural number $N = N(\epsilon)$ that, for any $i \in \mathbb{Z}$, there is at least one number $\varrho \in [i, i+N]$, for which (10.1) holds for all $k \in \mathbb{Z}$.* $\boxtimes$

Definition 10.1.3. *A function $\boldsymbol{f}(t)$ is almost periodic if*

1. *for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if t_1 and t_2 belong to the same interval of continuity and $|t_1 - t_2| < \delta$, then $\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\| < \epsilon$;*
2. *for any $\epsilon > 0$ there is a relatively dense set S of ϵ-almost periods such that if $\tau \in S$, then $\|\boldsymbol{f}(t+\tau) - \boldsymbol{f}(t)\| < \epsilon$ for all $t \in \mathbb{R}$ that satisfy $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$.*

$\boxtimes$

Let $\mathfrak{T}$ be a countable set of real numbers. $\mathfrak{T}$ includes positive and negative numbers with arbitrarily large absolute values. For any $\omega > 0$, the set $\{t \in \mathfrak{T} \mid |t| \leq \omega\}$ is finite. We denote all such sets as $\mathfrak{B}$. On $\mathfrak{B}$ we define a distance d_b as

$$d_b(\mathfrak{T}_1, \mathfrak{T}_2) \triangleq \inf_{\varphi} \sup_{t \in \mathfrak{T}_1} |\varphi(t) - t|$$

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 289−306, 2001.

where $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}$, $\varphi : \mathfrak{T}_1 \to \mathfrak{T}_2$ is a bijection. If $\mathfrak{T} \in \mathfrak{B}$ then for any $\tau \in \mathbb{R}$, the set $\mathfrak{T} + \tau$ is constructed as $\mathfrak{T} + \tau = \{t + \tau \mid \text{every } t \in \mathfrak{T}\}$. The space $(\mathfrak{B}, d_b)$ is complete.

Definition 10.1.4. $\mathfrak{T} \in \mathfrak{B}$ *is* Bohr almost periodic *if for any* $\epsilon > 0$ *there is such* $L(\epsilon) > 0$ *that one can find in any interval of length* $L(\epsilon)$ *such a number* τ *that* $d_b(\mathfrak{T}, \mathfrak{T} + \tau) < \epsilon$. ⊠

For any $i, j \in \mathbb{Z}$, let us denote $\tau_i^j \triangleq \tau_{i+j} - \tau_i$.

Definition 10.1.5. *The family of time sequences* $\{\tau_i^j\}$, $i, j \in \mathbb{Z}$, *is equipotentially almost periodic if for any* $\epsilon > 0$ *there is a relatively dense set of* ϵ*-almost periods, that are common to all sequences* $\{\tau_i^j\}$, $i, j \in \mathbb{Z}$. ⊠

10.2 Bohr Almost Periodic Linear Systems

Let us consider the following impulsive control system with impulses at fixed time:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t),\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x} + \boldsymbol{c}_k, \quad t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \qquad (10.2)}$$

where $A : C[\mathbb{R}_+, \mathbb{R}^{n\times n}]$ is Bohr almost periodic, $B_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{u}(t)$ and $\boldsymbol{c}_k$ are almost periodic. The sequence of time $\{\tau_i\}$ is such that $\{\tau_i^j\}$ is equipotentially almost periodic. We set a reference system as

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x},\ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x},\ t = \tau_k, \quad k \in \mathbb{Z}. \end{cases} \tag{10.3}$$

From the above assumptions we have the following Lemmas whose proofs can be found in [27].

Lemma 10.2.1. *For any* $\epsilon > 0$ *and any* $\theta > 0$ *there is such a real number* $\nu \in (0, \epsilon)$ *and relatively dense sets of real numbers* R, *and integers* Q *that the following relations hold for* $r \in R$ *and* $q \in Q$:

1. $\|\boldsymbol{u}(t + r) - \boldsymbol{u}(t)\| < \epsilon$ *for* $t \in \mathbb{R}$, $|t - \tau_i| > \epsilon$, $i \in \mathbb{Z}$;
2. $\|B_{k+q} - B_k\| < \epsilon$, $\|\boldsymbol{c}_{k+q} - \boldsymbol{c}_k\| < \epsilon$, $k \in \mathbb{Z}$;
3. $|\tau_k^q - r| < \nu$, $k \in \mathbb{Z}$.

⊠

Lemma 10.2.2. *If the conditions in Lemma 10.2.1 hold and the sum*

$$\$(t) = \int_0^t \boldsymbol{u}(s)ds + \sum_{0 \leq \tau_k < t} \boldsymbol{c}_k, \quad t > 0,$$

or

$$\$(t) = \int_0^t \boldsymbol{u}(s)ds + \sum_{t \le \tau_k < 0} \boldsymbol{c}_k, \quad t < 0$$

is bounded, then $\$(t)$ *is almost periodic.* ⊠

Lemma 10.2.3. *Let* $\Psi(t,s)$ *be the Cauchy matrix of the reference system (10.3) and assume that for* $a, b > 0$

$$\|\Psi(t,s)\| \le ae^{-b(t-s)}, \quad t \ge s$$

then for any $\epsilon > 0$, $t, s \in \mathbb{R}$, $|t-\tau_i| > \epsilon$, $|s-\tau_i| > \epsilon$, $i \in \mathbb{Z}$, *there is a relatively dense set of almost periods,* R, *such that for* $r \in R$ *we have*

$$\|\Psi(t+r, s+r) - \Psi(t,s)\| < \epsilon\alpha e^{-b(t-s)/2}$$

where $\alpha > 0$ *is a constant.* ⊠

Lemma 10.2.4. *If* $\{\tau_i^j\}$ *is equipotentially almost periodic, then for any* $i > 0$ *there is such a* $\varrho \in \mathbb{N}$ *that in any interval of the real axis of length* $L > 0$, *there are no more than* ϱ *terms of the sequence* $\{\tau_i\}$. ⊠

Lemma 10.2.5. *If* $\{\tau_k^1\}$ *is almost periodic then there exists the limit*

$$\lim_{T\to\infty} \frac{\mathfrak{N}(t, t+T)}{T} = p$$

uniformly with respect to $t \in \mathbb{R}$. ⊠

Let $\mu^{\max}(t)$ be the largest eigenvalue of the matrix

$$\frac{1}{2}[A(t) + A^*(t)]$$

and $\lambda_k^{\max}$ be the largest eigenvalue of the matrix

$$(I + B_k)^*(I + B_k)$$

and let us define

$$\gamma \triangleq \sup_t \mu^{\max}(t), \quad \alpha^2 \triangleq \max_k (\lambda_k^{\max})^2 .$$

Then it follows from the assumptions that for any $\epsilon > 0$

$$\|\Psi(t,s)\| \le Ke^{\beta(t-s)} \tag{10.4}$$

where $K = K(\epsilon) \ge 1$ and $\beta = \beta(\epsilon) = \epsilon + \gamma + p \ln \alpha$.

Theorem 10.2.1. *If* $\gamma + p\ln\alpha < 0$ *then system (10.2) has a unique almost periodic solution which is asymptotically stable.* ⊠

Proof. Let us choose $\boldsymbol{x}_1(t_0^+) = 0$ in (1.35), then we can construct the following solution of system (10.2):

$$\boldsymbol{x}_1(t) = \int_{-\infty}^{t} \Psi(t,s)\boldsymbol{u}(s)ds + \sum_{\tau_i<t} \Psi(t,\tau_i)\boldsymbol{c}_i. \tag{10.5}$$

First, let us check whether $\boldsymbol{x}_1(t)$ is bounded or not. Since $\{\tau_i^j\}$ is equipotentially almost periodic, it follows from Lemma 10.2.4 that there is such a constant $\xi > 0$ that

$$\tau_i^1 > \xi \text{ for all } i \in \mathbb{Z}. \tag{10.6}$$

Let us choose an $\epsilon \in (0, -\gamma - p\ln\alpha)$ such that $\beta = \beta(\epsilon) < 0$ and let $G \geq 0$ be

$$G = \sup_t \|\boldsymbol{u}(t)\| + \max_k \|\boldsymbol{c}_k\| \tag{10.7}$$

then for $t \in (\tau_k, \tau_{k+1}]$ we have

$$\begin{aligned}
\|\boldsymbol{x}_1(t)\| &\leq \int_{-\infty}^{t} \|\Psi(t,s)\|\|\boldsymbol{u}(s)\|ds + \sum_{\tau_i<t} \|\Psi(t,\tau_i)\|\|\boldsymbol{c}_i\| \\
&\leq \int_{-\infty}^{t} KGe^{\beta(t-s)}ds + \sum_{\tau_i<t} KGe^{\beta(t-\tau_i)} \qquad \Leftarrow (10.4) \\
&< KG\left(-\frac{1}{\beta} + \sum_{\tau_i<t} e^{\beta(t-\tau_i)}\right) \\
&= KG\left[-\frac{1}{\beta} + e^{\beta(t-\tau_k)} \sum_{\tau_i<t} \exp\left(\beta\sum_{j=i}^{k-1}(\tau_{j+1}-\tau_j)\right)\right] \\
&< KG\left(-\frac{1}{\beta} + e^{\beta(t-\tau_k)} \sum_{i=-\infty}^{k-1} e^{\beta\xi(k-1-i)}\right) \qquad \Leftarrow (10.6) \\
&= KG\left(-\frac{1}{\beta} + \frac{e^{\beta(t-\tau_k)}}{1-e^{\beta\xi}}\right) \\
&< KG\left(-\frac{1}{\beta} + \frac{1}{1-e^{\beta\xi}}\right) \\
&< \infty.
\end{aligned} \tag{10.8}$$

Therefore $\boldsymbol{x}_1(t)$ is bounded. We are then going to prove that $\boldsymbol{x}_1(t)$ is almost periodic. It follows form Lemmas 10.2.1 and 10.2.3 that

$$\begin{aligned}
\|\boldsymbol{x}_1(t+r) - \boldsymbol{x}_1(t)\| \leq \int_{-\infty}^{t} &\|\Psi(t+r,s+r)\boldsymbol{u}(s+r) - \Psi(t,s)\boldsymbol{u}(s)\| \\
&+ \sum_{\tau_i<t} \|\Psi(t+r,\tau_{i+q})\boldsymbol{c}_{i+q} - \Psi(t,\tau_i)\boldsymbol{c}_i\| < L(\epsilon)\epsilon
\end{aligned} \tag{10.9}$$

where $L(\epsilon)$ is a bounded positive function of ϵ. Therefore, $\boldsymbol{x}_1(t)$ is almost periodic. A solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ with $\boldsymbol{x}(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ of system (10.2) can be given by

$$\boldsymbol{x}(t) = \Psi(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Psi(t, s)\boldsymbol{u}(s)ds + \sum_{t_0 \le \tau_i < t} \Psi(t, \tau_i)\boldsymbol{c}_i. \qquad (10.10)$$

Given two distinct solutions $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ and $\boldsymbol{y}(t) = \boldsymbol{y}(t, \boldsymbol{y}_0)$, it follows from (10.4) that

$$\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\| \le K e^{\beta(t-t_0)} \|\boldsymbol{x}(t_0) - \boldsymbol{y}(t_0)\|, \qquad (10.11)$$

from which and in view of $\beta < 0$ we know that the almost periodic solution of system (10.2) is unique and asymptotically stable. ■

Let us consider the following impulsive control system with impulses at fixed time:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t, \boldsymbol{x}),\ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x} + \boldsymbol{c}_k(\boldsymbol{x}),\ \ t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \qquad (10.12)}$$

where $A : C[\mathbb{R}_+, \mathbb{R}^{n\times n}]$ is Bohr almost periodic, $B_k \in \mathbb{R}^{n\times n}$ is almost periodic.

$\boldsymbol{u}(t, \boldsymbol{x}), t \in \mathbb{R}$ is almost periodic in t and $\{\boldsymbol{c}_k(\boldsymbol{x})\}, k \in \mathbb{Z}$ is almost periodic in k and both properties are uniform in $\boldsymbol{x} \in \mathcal{S}_\rho$. Let us assume that $\boldsymbol{u}(t, \boldsymbol{x})$ and $\boldsymbol{c}_k(\boldsymbol{x})$ satisfy the following Lipschitz condition

$$\|\boldsymbol{u}(t, \boldsymbol{x}) - \boldsymbol{u}(t, \boldsymbol{y})\| + \|\boldsymbol{c}_k(\boldsymbol{x}) - \boldsymbol{c}_k(\boldsymbol{y})\| \le L_1 \|\boldsymbol{x} - \boldsymbol{y}\| \qquad (10.13)$$

and

$$\sup_{t\in\mathbb{R}, \boldsymbol{x}\in\mathcal{S}_\rho} \|\boldsymbol{u}(t, \boldsymbol{x})\| + \sup_{t\in\mathbb{R}, \boldsymbol{x}\in\mathcal{S}_\rho} \|\boldsymbol{c}_k(\boldsymbol{x})\| \triangleq L_2 < \infty. \qquad (10.14)$$

The sequence of time $\{\tau_i\}$ is such that $\{\tau_i^j\}$ is equipotentially almost periodic and

$$\inf_k \tau_k^1 = \xi > 0.$$

We set a reference system as

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x},\ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x},\ t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \qquad (10.15)$$

whose Cauchy matrix is $\Psi(t, s)$. From the above assumptions we have the following Lemmas whose proofs can be found in [27].

Lemma 10.2.6. *Let $\boldsymbol{f}(t)$ be an almost periodic function and the sequence $\{\tau_i\}$ is such that $\{\tau_i^j\}$ is equipotentially almost periodic and*

$$\inf_k \tau_k^1 = \xi > 0,$$

then $\{\boldsymbol{f}(\tau_k)\}$ is an almost periodic sequence. ⊠

Lemma 10.2.7. *Let $\boldsymbol{f} : \mathbb{R} \to \Omega$, $\Omega \subset \mathbb{R}^n$, be an almost periodic function and $\boldsymbol{g} : \Omega \to \mathbb{R}^n$ be a uniformly continuous function, then the function $\boldsymbol{g}(\boldsymbol{f}(t)), t \in \mathbb{R}$ is almost periodic.* ⊠

Similarly, it follows from the assumptions that for any $\epsilon > 0$

$$\|\Psi(t,s)\| \leq Ke^{\beta(t-s)} \tag{10.16}$$

where $K = K(\epsilon) \geq 1$.

Let Ξ be the space of all almost periodic functions with discontinuities at the points of $\{\tau_k\}$. For any $\boldsymbol{f}(t) \in \Xi$ we define its norm on Ξ as

$$\|\boldsymbol{f}(t)\|_\Xi \triangleq \sup_{t\in\mathbb{R}} \|\boldsymbol{f}(t)\|. \tag{10.17}$$

Let $\Omega \subset \Xi$ such that for any $\boldsymbol{x}(t) \in \Omega$ we have $\|\boldsymbol{x}(t)\|_\Xi < \rho$. For $\boldsymbol{x}(t) \in \Xi$ we defined an operator $\mathcal{O}$ as follows:

$$\begin{aligned}\mathcal{O}(\boldsymbol{x}(t)) \triangleq &\int_{-\infty}^{t} \Psi(t,s)\boldsymbol{u}(s,\boldsymbol{x}(s))ds \\ &+ \sum_{\tau_k<t} \Psi(t,\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)).\end{aligned} \tag{10.18}$$

Let us assume that $\|\boldsymbol{x}(t)\|_\Xi < \rho$. Since $\boldsymbol{x}(t)$ is almost periodic, it follows from Lemma 10.2.6 that the sequence $\{\boldsymbol{x}(\tau_k)\}$ is almost periodic. Therefore the sequence $\{\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\}$ is almost periodic. It follows from Lemma 10.2.7 that the function $\boldsymbol{u}(t,\boldsymbol{x}(t))$ is almost periodic. It follows from Lemmas 10.2.1 and 10.2.3 that if $\boldsymbol{x}(t) \in \Omega$, then there is a relatively dense set R of ϵ-almost periods of $\boldsymbol{x}(t)$ such that, for $r \in R$, $t \in \mathbb{R}$, $|t-\tau_k| > \epsilon$ and $k \in \mathbb{Z}$ we have

$$\begin{aligned}&\|\mathcal{O}(\boldsymbol{x}(t+r)) - \mathcal{O}(\boldsymbol{x}(t))\| \\ &\leq \int_{-\infty}^{t} \|\Psi(t+r,s+r)\boldsymbol{u}(s+r,\boldsymbol{x}(s+r)) - \Psi(t,s)\boldsymbol{u}(s,\boldsymbol{x}(s))\|ds \\ &+ \sum_{\tau_k<t} \|\Psi(t+r,\tau_{k+q})\boldsymbol{c}_{k+q}(\boldsymbol{x}(\tau_{k+q})) - \Psi(t,\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\| \\ &\leq L_3(\epsilon)\epsilon\end{aligned} \tag{10.19}$$

where $L_3(\epsilon)$ is bounded with respect to ϵ. Therefore $\mathcal{O}(\boldsymbol{x}(t)) \in \Xi$. Furthermore, given $\|\boldsymbol{x}(t)\|_\Xi < \rho$ and $t \in (\tau_k, \tau_{k+1}]$ we have

$$\begin{aligned}
\|\mathcal{O}(\boldsymbol{x}(t))\|_{\Xi} &\leq \int_{-\infty}^{t} \|\Psi(t,s)\| \sup_{t\in\mathbb{R}} \|\boldsymbol{u}(s,\boldsymbol{x}(s))\| ds \\
&\quad + \sum_{\tau_k<t} \|\Psi(t,\tau_k)\| \sup_{t\in\mathbb{R}} \|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\| \\
&\leq \int_{-\infty}^{t} K e^{\beta(t-s)} L_2 ds + \sum_{\tau_k<t} K e^{\beta(t-\tau_k)} L_2 \\
&< KL_2 \left(-\frac{1}{\beta} + \sum_{\tau_i<t} e^{\beta(t-\tau_i)} \right) \\
&= KL_2 \left[-\frac{1}{\beta} + e^{\beta(t-\tau_k)} \sum_{\tau_i<t} \exp\left(\beta \sum_{j=i}^{k-1} (\tau_{j+1}-\tau_j) \right) \right] \\
&< KL_2 \left(-\frac{1}{\beta} + e^{\beta(t-\tau_k)} \sum_{i=-\infty}^{k-1} e^{\beta\xi(k-1-i)} \right) \qquad \Leftarrow (10.6) \\
&= KL_2 \left(-\frac{1}{\beta} + \frac{e^{\beta(t-\tau_k)}}{1-e^{\beta\xi}} \right) \\
&< KL_2 \left(-\frac{1}{\beta} + \frac{1}{1-e^{\beta\xi}} \right). \qquad (10.20)
\end{aligned}$$

If $KL_2\left(-\frac{1}{\beta}+\frac{1}{1-e^{\beta\xi}}\right) < \rho$ then we have $\mathcal{O}(\Omega) \subseteq \Omega$. Let us define

$$L_4 \triangleq -\frac{1}{\beta} + \frac{1}{1-e^{\beta\xi}}.$$

Given any two functions $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ we have

$$\begin{aligned}
\|\mathcal{O}(\boldsymbol{x}(t)) - \mathcal{O}(\boldsymbol{y}(t))\|_{\Xi} &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|\Psi(t,s)\| \|\boldsymbol{u}(s,\boldsymbol{x}(s)) - \boldsymbol{u}(s,\boldsymbol{y}(s))\| ds \\
&\quad + \sum_{\tau_k<t} \|\Psi(t,\tau_k)\| \|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)) - \boldsymbol{c}_k(\boldsymbol{y}(\tau_k))\| \\
&\leq KL_1L_4 \|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|_{\Xi}. \qquad (10.21)
\end{aligned}$$

If $KL_1L_4 < 1$ then $\mathcal{O}$ is a contraction. Then from fixed point theorem we know that there is an almost periodic solution of system (10.12). A solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ with $\boldsymbol{x}(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ of system (10.12) can be given by

$$\begin{aligned}
\boldsymbol{x}(t) &= \Psi(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Psi(t,s)\boldsymbol{u}(s,\boldsymbol{x}(s)) ds \\
&\quad + \sum_{t_0\leq\tau_k<t} \Psi(t,\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)). \qquad (10.22)
\end{aligned}$$

Then for any two solutions of system (10.12) $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ and $\boldsymbol{y}(t) = \boldsymbol{y}(t, \boldsymbol{y}_0)$ we have

$$\begin{aligned}\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\| \le\ & \|\Psi(t,t_0)\|\|\boldsymbol{x}_0-\boldsymbol{y}_0\| \\ & + \int_{t_0}^{t} \|\Psi(t,s)\|\|\boldsymbol{u}(s,\boldsymbol{x}(s))-\boldsymbol{u}(s,\boldsymbol{y}(s))\|ds \\ & + \sum_{t_0\le\tau_k<t} \|\Psi(t,\tau_k)\|\|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)-\boldsymbol{c}_k(\boldsymbol{y}(\tau_k)\| \\ \le\ & Ke^{\beta(t-t_0)}\|\boldsymbol{x}_0-\boldsymbol{y}_0\| + \int_{t_0}^{t} Ke^{\beta(t-s)}L_1\|\boldsymbol{x}(s)-\boldsymbol{y}(s)\|ds \\ & + \sum_{t_0\le\tau_k<t} Ke^{\beta(t-\tau_k)}L_1\|\boldsymbol{x}(\tau_k)-\boldsymbol{y}(\tau_k)\| \end{aligned} \tag{10.23}$$

from which and using the transformation $u(t) \triangleq \|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|e^{-\beta t}$ we have

$$u(t) \le \underbrace{Ku(t_0)}_{c} + \int_{t_0}^{t} \underbrace{KL_1}_{v(s)} u(s)ds + \sum_{t_0\le\tau_k<t} \underbrace{KL_1}_{b_k} u(\tau_k) \tag{10.24}$$

which has the same form as (1.43). Then it follows from Lemma 1.7.1 we have

$$u(t) \le Ku(t_0) \prod_{t_0\le\tau_k<t} (1+KL_1)e^{KL_1(t-t_0)}; \tag{10.25}$$

namely,

$$\begin{aligned}\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\| &\le K\|\boldsymbol{x}(t_0)-\boldsymbol{y}(t_0)\| \prod_{t_0\le\tau_k<t} (1+KL_1)e^{(\beta+KL_1)(t-t_0)} \\ &= K\|\boldsymbol{x}(t_0)-\boldsymbol{y}(t_0)\|(1+KL_1)^{\mathfrak{N}(t_0,t)}e^{(\beta+KL_1)(t-t_0)}. \end{aligned} \tag{10.26}$$

Let us assume that $\mathfrak{N}(t_0,t) = \varpi(t_0,t)(t-t_0)$ then we have

$$\begin{aligned}\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\| \le\ & K\|\boldsymbol{x}(t_0)-\boldsymbol{y}(t_0)\| \\ & \times \exp\left[\left(\beta+KL_1+\frac{\ln(1+KL_1)}{\varpi(t_0,t)}\right)(t-t_0)\right] \end{aligned} \tag{10.27}$$

from which we know that if $\beta+KL_1+\frac{\ln(1+KL_1)}{\varpi(t_0,t)} < 0$ then solutions of system (10.12) are asymptotically stable. Therefore its almost periodic solution is unique and asymptotically stable. This finishes the proof of the following theorem.

Theorem 10.2.2. *Assume that*

1. $KL_2\left(-\frac{1}{\beta}+\frac{1}{1-e^{\beta\xi}}\right) < \rho$;
2. $KL_1L_4 < 1$;

3. $\beta+KL_1+\frac{\ln(1+KL_1)}{\varpi(t_0,t)} < 0$ *for big enough* $t-t_0$, *where* $\mathfrak{N}(t_0,t) = \varpi(t_0,t)(t-t_0)$.

Then system (10.12) has a unique almost periodic solution which is asymptotically stable. ⊠

10.3 *T*-Periodic Linear System with Almost Periodic Control

In this section we study T-periodic linear systems that controlled by almost periodic control signals.

10.3.1 First-Order Case

Let us consider the following first-order impulsive control system with impulses at fixed time:

$$\begin{cases} \dot{x} = \mu x + u(t), \; t \neq \tau_k, \\ \Delta x = c_k, \qquad\qquad t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \tag{10.28}$$

where $\mu \in \mathbb{C}$, $u(t)$ and $\{c_k\}$ are almost periodic and there is an integer ϱ such that $\tau_{k+\varrho} = \tau_k + T$ for $k \in \mathbb{Z}$. Then we have the following Theorem.

Theorem 10.3.1. *If system (10.28) has a bounded solution, then this solution is almost periodic.* ⊠

Proof. A solution $x(t) = x(t, x_0)$ with $x(0, x_0) = 0$ of system (10.28) is given by

$$x(t) = e^{\mu t}\left(x_0 + \int_0^t e^{-\mu s}u(s)ds + \sum_{0\le\tau_k<t} e^{-\mu\tau_k}c_k\right), \; t > 0,$$
$$x(t) = e^{\mu t}\left(x_0 + \int_0^t e^{-\mu s}u(s)ds - \sum_{t\le\tau_k<0} e^{-\mu\tau_k}c_k\right), \; t < 0. \tag{10.29}$$

We then study the following three cases.

1. Re $\mu > 0$. In this case we know that

$$\lim_{t\to\infty} |e^{\mu t}| = \infty.$$

   Therefore, in order to have a bounded solution, the initial condition should be chosen as

$$x_0 = -\int_0^{\infty} e^{-\mu s} u(s)ds - \sum_{\tau_k \geq 0} e^{-\mu \tau_k} c_k \tag{10.30}$$

which is converge. Let us substitute x_0 into the first equation in (10.29) and get

$$x(t) = -\int_t^{\infty} e^{\mu(t-s)} u(s)ds - \sum_{\tau_k \geq t} e^{\mu(t-\tau_k)} c_k. \tag{10.31}$$

It follows from Lemma 10.2.1 that there is such an almost ϵ-period $r = r(\epsilon)$ that

$$\begin{aligned}|x(t+r) - x(t)| &\leq \int_t^{\infty} e^{\mu(t-s)} |u(s+r) - u(s)| ds \\ &\quad + \sum_{\tau_k \geq t} e^{\mu(t-\tau_k)} |c_{k+\varrho} - c_k| \\ &< L(\epsilon)\epsilon \end{aligned} \tag{10.32}$$

where $L(\epsilon)$ is bounded. Therefore, we have proved that $x(t)$ is an almost periodic solution.

2. Re $\mu < 0$. In this case we know that

$$\lim_{t \to -\infty} |e^{\mu t}| = \infty.$$

Therefore, in order to have a bounded solution, the initial condition should be chosen as

$$x_0 = \int_{-\infty}^{0} e^{-\mu s} u(s)ds + \sum_{\tau_k < 0} e^{-\mu \tau_k} c_k. \tag{10.33}$$

which is converge. Let us substitute x_0 into the second equation in (10.29) and get

$$x(t) = \int_{-\infty}^{t} e^{\mu(t-s)} u(s)ds + \sum_{\tau_k < t} e^{\mu(t-\tau_k)} c_k. \tag{10.34}$$

It follows from Lemma 10.2.1 that there is such an almost ϵ-period $r = r(\epsilon)$ that

$$\begin{aligned}|x(t+r) - x(t)| &\leq \int_{-\infty}^{t} e^{\mu(t-s)} |u(s+r) - u(s)| ds \\ &\quad + \sum_{\tau_k < t} e^{\mu(t-\tau_k)} |c_{k+\varrho} - c_k| \\ &< L_1(\epsilon)\epsilon \end{aligned} \tag{10.35}$$

where $L_1(\epsilon)$ is bounded. Therefore, we have proved that $x(t)$ is an almost periodic solution.

3. Re $\mu = 0$. Let us denote $\mu = i\omega$. Let us first study the case when $t > 0$. From the first equation of (10.29) we know that a bounded solution is given by

$$x(t) = e^{-i\omega t}\left(x_0 + \int_0^t e^{-i\omega s}u(s)ds + \sum_{0\leq\tau_k<t} e^{-i\omega\tau_k}c_k\right) \quad (10.36)$$

from which we know that the sum

$$\$(t) \triangleq \int_0^t e^{-i\omega s}u(s)ds + \sum_{0\leq\tau_k<t} e^{-i\omega\tau_k}c_k \quad (10.37)$$

is bounded. It is also clear that $e^{-i\omega s}u(s)$ and $\{e^{-i\omega\tau_k}c_k\}$ are almost periodic. It follows from Lemma 10.2.2 that $\$(t)$ is almost periodic. Therefore, any a solution of system (10.28) is almost periodic.

■

10.3.2 Multi-Dimensional Cases

Let us consider the following impulsive control system with impulses at fixed time:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t),\ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x} + \boldsymbol{c}_k, \quad t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \quad (10.38)}$$

where $A : C[\mathbb{R}_+, \mathbb{R}^{n\times n}]$ is T-periodic, $B_k \in \mathbb{R}^{n\times n}$ is a constant matrix, $\boldsymbol{u}(t)$ and $\boldsymbol{c}_k$ are almost periodic and there is an integer ϱ such that $\tau_{k+\varrho} = \tau_k + T$ for $k \in \mathbb{Z}$. We need the following reference system:

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x},\ t \neq \tau_k, \\ \Delta\boldsymbol{x} = B_k\boldsymbol{x},\ t = \tau_k, \quad k \in \mathbb{Z}. \end{cases} \quad (10.39)$$

Let M be a monodromy matrix of system (10.39) and let

$$\Lambda = \frac{1}{T}\ln M \quad (10.40)$$

then we have the following Theorems.

Theorem 10.3.2. *Assume that Λ has no eigenvalues on the imaginary axis of the s-plane, then the impulsive control system (10.38) has*

1. *a unique almost periodic solution ;*

2. *a unique almost periodic solution that is asymptotically stable if all eigenvalues of Λ are in the left-half s-plane.*

⊠

Proof. Let us first prove conclusion 1. Without loss of generality, let us assume that $\Lambda = \text{diag}(R, L)$ where $R \in \mathbb{R}^{m\times m}$ and $L \in \mathbb{R}^{(n-m)\times(n-m)}$ such that all eigenvalues of L and R are in the left-hand side and the right-hand side of the s-plane, respectively. Let $\hat{\Upsilon}(t)$ be an $n \times n$ matrix defined as

$$\hat{\Upsilon}(t) \triangleq \begin{cases} -\text{diag}(e^{Rt}, 0), & \text{for } t < 0, \\ \text{diag}(0, e^{Lt}), & \text{for } t > 0 \end{cases} \tag{10.41}$$

then we have the following properties:

$$\hat{\Upsilon}(0^+) - \hat{\Upsilon}(0^-) = I \tag{10.42}$$

and for $t \neq 0$ we have

$$\frac{\partial \hat{\Upsilon}(t)}{\partial t} = \Lambda \hat{\Upsilon}(t). \tag{10.43}$$

Given $a, b > 0$, then we have

$$\|\hat{\Upsilon}(t)\| \leq a e^{-b|t|}. \tag{10.44}$$

Let us apply a coordinate transform $\boldsymbol{x} = X(t)e^{-\Lambda t}\boldsymbol{y} \triangleq \Theta(t)\boldsymbol{y}$ to system (10.38) then we have

$$\begin{cases} \dot{\boldsymbol{y}} = \Lambda \boldsymbol{y} + \boldsymbol{v}(t), & t = \tau_k, \\ \Delta \boldsymbol{y} = \Theta^{-1}(\tau_k^+) B_k \Theta(\tau_k) \boldsymbol{y} + \Theta^{-1}(\tau_k^+)\boldsymbol{c}_k, & t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \tag{10.45}$$

where $\boldsymbol{v}(t) = \Theta^{-1}(t)\boldsymbol{u}(t)$ and denote V_k as

$$V_k \triangleq \Theta^{-1}(\tau_k^+)\boldsymbol{c}_k \tag{10.46}$$

we then have the following claim:
Claim 10.3.2:

$$\boldsymbol{y}_1(t) \triangleq \int_{-\infty}^{\infty} \hat{\Upsilon}(t-s)\boldsymbol{v}(s)ds + \sum_{k=-\infty}^{\infty} \hat{\Upsilon}(t-\tau_k)V_k \tag{10.47}$$

is a solution of (10.45).

To prove that Claim 10.3.2 is true, we first show that $\boldsymbol{y}_1(t)$ is bounded.

$$\begin{aligned}
\|\boldsymbol{y}_1(t)\| &\le \int_{-\infty}^{\infty} \|\hat{\Upsilon}(t-s)\|\|\boldsymbol{v}(s)\|ds + \sum_{k=-\infty}^{\infty} \|\hat{\Upsilon}(t-\tau_k)\|\|V_k\| \\
&\le a \sup_{t=-\infty}^{\infty} \|\boldsymbol{v}(t)\| \int_{-\infty}^{\infty} e^{-b|t-s|}ds \\
&\quad + a \max_{k=-\infty}^{\infty} \|V_k\| \sum_{k=-\infty}^{\infty} e^{-b|t-\tau_k|} \qquad\qquad \Leftarrow (10.44) \\
&\le a \sup_{t=-\infty}^{\infty} \|\boldsymbol{v}(t)\| \left. \frac{\operatorname{sgn}(t-s)}{b} e^{-b|t-s|} \right|_{s=-\infty}^{\infty} \\
&\quad + a \max_{k=-\infty}^{\infty} \|V_k\| \sum_{k=-\infty}^{\infty} e^{-b|k|\varpi} \\
&\le a \max\left(\frac{2}{b}, \frac{2}{1-e^{-b\varpi}}\right) \big(\sup_{t=-\infty}^{\infty} \|\boldsymbol{v}(t)\| + \max_{k=-\infty}^{\infty} \|V_k\|\big) \\
&< \infty
\end{aligned} \tag{10.48}$$

where

$$\varpi = \min_{k\in\mathbb{Z}} (\tau_{k+1} - \tau_k). \tag{10.49}$$

Then it follows (10.42) and (10.43) that

$$\begin{aligned}
\dot{\boldsymbol{y}}_1 &= [\hat{\Upsilon}(0^+) - \hat{\Upsilon}(0^-)]\boldsymbol{v}(t) + \int_{-\infty}^{t} \Lambda\hat{\Upsilon}(t-s)\boldsymbol{v}(s)ds \\
&\quad + \int_{t}^{\infty} \Lambda\hat{\Upsilon}(t-s)\boldsymbol{v}(s)ds + \Lambda \sum_{k=-\infty}^{\infty} \hat{\Upsilon}(t-\tau_k)V_k \\
&= \Lambda\boldsymbol{y}_1(t) + \boldsymbol{v}(t), \qquad t \neq \tau_k \\
\Delta\boldsymbol{y}_1 &= \sum_{i=-\infty}^{\infty} \hat{\Upsilon}(\tau_k - \tau_i^+)V_i - \sum_{i=-\infty}^{\infty} \hat{\Upsilon}(\tau_k - \tau_i)V_i \\
&= V_k, , \qquad t = \tau_k,
\end{aligned} \tag{10.50}$$

which shows that $\boldsymbol{y}_1$ is a solution of system (10.45) and Claim 10.3.2 is true.

For an $\epsilon > 0$, it follows Lemma 10.2.1 that there is such an ϵ-almost period r that

$$\begin{aligned}\boldsymbol{y}_1(t+r)-\boldsymbol{y}_1(t) &= \int_{-\infty}^{\infty}\hat{\Upsilon}(t+r-s)\boldsymbol{v}(s)ds+\sum_{i=-\infty}^{\infty}\hat{\Upsilon}(t+r-\tau_i)V_i\\ &\quad -\int_{-\infty}^{\infty}\hat{\Upsilon}(t-s)\boldsymbol{v}(s)ds-\sum_{i=-\infty}^{\infty}\hat{\Upsilon}(t-\tau_i)V_i\\ &= \int_{-\infty}^{\infty}\hat{\Upsilon}(t-s)[\boldsymbol{v}(s+\rho)-\boldsymbol{v}(s)]ds\\ &\quad +\sum_{i=-\infty}^{\infty}\hat{\Upsilon}(t-\tau_i)(V_{i+\varrho}-V_i)\end{aligned}\tag{10.51}$$

and

$$\begin{aligned}\|\boldsymbol{y}_1(t+r)-\boldsymbol{y}_1(t)\| &\le \int_{-\infty}^{\infty} ae^{-b|t-s|}\|\boldsymbol{v}(s+r)-\boldsymbol{v}(s)\|ds\\ &\quad +\sum_{i=-\infty}^{\infty} ae^{-b|t-\tau_i|}\|V_{i+\varrho}-V_i\| < K(\epsilon)\epsilon\end{aligned}\tag{10.52}$$

where $K(\epsilon)$ is bounded. This proves that $\boldsymbol{y}_1(t)$ is almost periodic. Let us suppose that $\boldsymbol{y}_2(t)$ is an almost periodic solution, which is different from $\boldsymbol{y}_1(t)$, of system (10.45). Then $\boldsymbol{y}_2(t)-\boldsymbol{y}_1(t)$ is an almost periodic solution of the following system:

$$\dot{\boldsymbol{y}} = \Lambda\boldsymbol{y}. \tag{10.53}$$

Since Λ has no eigenvalues on the imaginary axis of the s-plane , system (10.53) has no non-trivial almost periodic solutions. This leads to a contradiction. Therefore, $\boldsymbol{y}_1(t)$ is the unique almost periodic solution of system (10.45). The system (10.38) has the following unique almost periodic solution:

$$\begin{aligned}\boldsymbol{x}_1(t) &= \int_{-\infty}^{\infty}\Theta(t)\hat{\Upsilon}(t-s)\Theta^{-1}(s)\boldsymbol{u}(s)ds\\ &\quad +\sum_{i=-\infty}^{\infty}\Theta(t)\hat{\Upsilon}(t-\tau_i)\Theta^{-1}(\tau_i)\boldsymbol{c}_i.\end{aligned}\tag{10.54}$$

We then prove conclusion 2. Since the difference between two solutions of system (10.45) is a solution of (10.53), we know that if all eigenvalues of Λ are in the left half s-plane then every solution of (10.45) asymptotically approaches the unique almost periodic solution. Therefore, the unique almost periodic solution of system (10.45) is asymptotically stable. Furthermore, we know that the unique almost periodic solution of system (10.38) is also asymptotically stable. ■

Theorem 10.3.3. *If system (10.38) has a bounded solution, then this solution is almost periodic.* ⊠

Proof. Let us first consider a bounded solution of system (10.45). We can always use a transformation to make system (10.45) into the following form:

$$\begin{aligned}
\dot{y}_1 &= \lambda_1 y_1 + b_{12} y_2 + \cdots + b_{1n} y_n + v_1(t) \\
\dot{y}_2 &= \lambda_2 y_2 + b_{23} y_3 + \cdots + b_{2n} y_n + v_2(t) \\
&\vdots \\
\dot{y}_{n-1} &= \lambda_{n-1} y_{n-1} + b_{n-1,n} y_n + v_{n-1}(t) \\
\dot{y}_n &= \lambda_n y_n + v_n(t), \qquad t \neq \tau_k, \\
\Delta y_1 &= V_{k,1}, \\
&\vdots \\
\Delta y_n &= V_{k,n}, \qquad t = \tau_k, \quad k \in \mathbb{Z}.
\end{aligned} \tag{10.55}$$

We then solve (10.55) from bottom to up and it follows from Theorem 10.3.1 that the bounded solution of system (10.45) is almost periodic. It follows that the bounded solution of system (10.38) is also almost periodic. ■

Let us study the following almost-periodic impulsive control system:

$$\boxed{\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + \boldsymbol{u}(t, \boldsymbol{x}), \ t \neq \tau_k, \\ \Delta \boldsymbol{x} = B_k \boldsymbol{x} + \boldsymbol{c}_k(\boldsymbol{x}), \ \ t = \tau_k, \quad k \in \mathbb{Z} \end{cases} \qquad (10.56)}$$

where $A : C[\mathbb{R}_+, \mathbb{R}^{n\times n}]$ is T-periodic, $B_k \in \mathbb{R}^{n\times n}$ is a constant matrix, $\boldsymbol{u}(t, \boldsymbol{x})$ is almost periodic in t and $\boldsymbol{c}_k(\boldsymbol{x})$ is almost periodic in k uniformly with respect to $\boldsymbol{x}$. There is an integer ϱ such that $\tau_{k+\varrho} = \tau_k + T$ for $k \in \mathbb{Z}$.

Let Ξ be the space of all almost periodic functions with discontinuities at the points $\{\tau_k\}$. For any $\boldsymbol{f}(t) \in \Xi$ we define its norm on Ξ as

$$\|\boldsymbol{f}(t)\|_\Xi \triangleq \sup_{t\in\mathbb{R}} \|\boldsymbol{f}(t)\|. \tag{10.57}$$

Let $\Omega \subset \Xi$ such that for any $\boldsymbol{x}(t) \in \Omega$ we have $\|\boldsymbol{x}(t)\|_\Xi < \rho$. For $\boldsymbol{x}(t) \in \Xi$ we define an operator $\mathcal{O}$ as follows:

$$\begin{aligned}
\mathcal{O}(\boldsymbol{x}(t)) \triangleq & \int_{-\infty}^{t} \Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\boldsymbol{u}(s, \boldsymbol{x}(s))ds \\
&+ \sum_{\tau_k<t} \Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)).
\end{aligned} \tag{10.58}$$

Let us assume that $\|\boldsymbol{x}(t)\|_\Xi < \rho$. Since $\boldsymbol{x}(t)$ is almost periodic, it follows from Lemma 10.2.6 that the sequence $\{\boldsymbol{x}(\tau_k)\}$ is almost periodic. Therefore the sequence $\{\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\}$ is almost periodic. It follows from Lemma 10.2.7 that the function $\boldsymbol{u}(t, \boldsymbol{x}(t))$ is almost periodic. It follows from Lemmas 10.2.1

and 10.2.3 that if $\boldsymbol{x}(t) \in \Omega$, then there is a relatively dense set R of ϵ-almost periods of $\boldsymbol{x}(t)$ such that, for $r \in R$, $t \in \mathbb{R}$, $|t-\tau_k| > \epsilon$ and $k \in \mathbb{Z}$ we have

$$\begin{aligned}
&\|\mathcal{O}(\boldsymbol{x}(t+r)) - \mathcal{O}(\boldsymbol{x}(t))\| \\
&\le \int_{-\infty}^{t} \|\Theta(t+r)\tilde{\Upsilon}(t-s)\Theta^{-1}(s+r)\boldsymbol{u}(s+r,\boldsymbol{x}(s+r)) \\
&-\Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\boldsymbol{u}(s,\boldsymbol{x}(s))\|ds \\
&+\sum_{\tau_k<t} \|\Theta(t+r)\tilde{\Upsilon}(t+r-\tau_{k+\varrho})\Theta^{-1}(\tau_{k+\varrho})\boldsymbol{c}_{k+\varrho}(\boldsymbol{x}(\tau_{k+\varrho})) \\
&-\Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\| \\
&\le L_3(\epsilon)\epsilon
\end{aligned} \tag{10.59}$$

where $L_3(\epsilon)$ is bounded with respect to ϵ. Therefore $\mathcal{O}(\boldsymbol{x}(t)) \in \Xi$. Furthermore, given $\|\boldsymbol{x}(t)\|_{\Xi} < \rho$ and $t \in (\tau_k, \tau_{k+1}]$ we have

$$\begin{aligned}
\|\mathcal{O}(\boldsymbol{x}(t))\|_{\Xi} &\le \int_{-\infty}^{t} \|\Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\| \sup_{t\in\mathbb{R}} \|\boldsymbol{u}(s,\boldsymbol{x}(s))\|ds \\
&\quad + \sum_{\tau_k<t} \|\Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\| \sup_{t\in\mathbb{R}} \|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))\| \\
&\le \int_{-\infty}^{t} Ke^{\beta(t-s)}L_2 ds + \sum_{\tau_k<t} Ke^{\beta(t-\tau_k)}L_2 \\
&< KL_2\left(-\frac{1}{\beta} + \sum_{\tau_i<t} e^{\beta(t-\tau_i)}\right) \\
&= KL_2\left[-\frac{1}{\beta} + e^{\beta(t-\tau_k)}\sum_{\tau_i<t}\exp\left(\beta\sum_{j=i}^{k-1}(\tau_{j+1}-\tau_j)\right)\right] \\
&< KL_2\left(-\frac{1}{\beta} + e^{\beta(t-\tau_k)}\sum_{i=-\infty}^{k-1} e^{\beta\xi(k-1-i)}\right) \qquad \Leftarrow (10.6) \\
&= KL_2\left(-\frac{1}{\beta} + \frac{e^{\beta(t-\tau_k)}}{1-e^{\beta\xi}}\right) \\
&< KL_2\left(-\frac{1}{\beta} + \frac{1}{1-e^{\beta\xi}}\right).
\end{aligned} \tag{10.60}$$

If $KL_2\left(-\frac{1}{\beta}+\frac{1}{1-e^{\beta\xi}}\right) < \rho$ then we have $\mathcal{O}(\Omega) \subseteq \Omega$. Let us define $L_4 \triangleq -\frac{1}{\beta}+\frac{1}{1-e^{\beta\xi}}$.

Given any two functions $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ we have

$$
\begin{aligned}
&\|\mathcal{O}(\boldsymbol{x}(t))-\mathcal{O}(\boldsymbol{y}(t))\|_{\Xi}\\
&\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\|\Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\|\|\boldsymbol{u}(s,\boldsymbol{x}(s))-\boldsymbol{u}(s,\boldsymbol{y}(s))\|ds\\
&\quad+\sum_{\tau_k<t}\|\Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\|\|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k))-\boldsymbol{c}_k(\boldsymbol{y}(\tau_k))\|\\
&\le KL_1L_4\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|_{\Xi}.
\end{aligned}
\tag{10.61}
$$

If $KL_1L_4 < 1$ then $\mathcal{O}$ is a contraction. Then from fixed point theorem we know that there is a almost periodic solution of system (10.56). A solution $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ with $\boldsymbol{x}(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ of system (10.56) can be given by

$$
\begin{aligned}
\boldsymbol{x}(t)&=\Theta(t)\tilde{\Upsilon}(t-t_0)\Theta^{-1}(t_0)\boldsymbol{x}_0+\int_{t_0}^{t}\Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\boldsymbol{u}(s,\boldsymbol{x}(s))ds\\
&\quad+\sum_{t_0\le\tau_k<t}\Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)).
\end{aligned}
\tag{10.62}
$$

Then for any two solutions of system (10.56) $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0)$ and $\boldsymbol{y}(t) = \boldsymbol{y}(t, \boldsymbol{y}_0)$ we have

$$
\begin{aligned}
\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|&\le\|\Theta(t)\tilde{\Upsilon}(t-t_0)\Theta^{-1}(t_0)\|\|\boldsymbol{x}_0-\boldsymbol{y}_0\|\\
&\quad+\int_{t_0}^{t}\|\Theta(t)\tilde{\Upsilon}(t-s)\Theta^{-1}(s)\|\|\boldsymbol{u}(s,\boldsymbol{x}(s))-\boldsymbol{u}(s,\boldsymbol{y}(s))\|ds\\
&\quad+\sum_{t_0\le\tau_k<t}\|\Theta(t)\tilde{\Upsilon}(t-\tau_k)\Theta^{-1}(\tau_k)\|\|\boldsymbol{c}_k(\boldsymbol{x}(\tau_k)-\boldsymbol{c}_k(\boldsymbol{y}(\tau_k)\|\\
&\le Ke^{\beta(t-t_0)}\|\boldsymbol{x}_0-\boldsymbol{y}_0\|+\int_{t_0}^{t}Ke^{\beta(t-s)}L_1\|\boldsymbol{x}(s)-\boldsymbol{y}(s)\|ds\\
&\quad+\sum_{t_0\le\tau_k<t}Ke^{\beta(t-\tau_k)}L_1\|\boldsymbol{x}(\tau_k)-\boldsymbol{y}(\tau_k)\|
\end{aligned}
\tag{10.63}
$$

from which and using the transformation $u(t) \triangleq \|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|e^{-\beta t}$ we have

$$
u(t)\le\underbrace{Ku(t_0)}_{c}+\int_{t_0}^{t}\underbrace{KL_1}_{v(s)}u(s)ds+\sum_{t_0\le\tau_k<t}\underbrace{KL_1}_{b_k}u(\tau_k)
\tag{10.64}
$$

which has the same form as (1.43). Then it follows from Lemma 1.7.1 that

$$
u(t)\le Ku(t_0)\prod_{t_0\le\tau_k<t}(1+KL_1)e^{KL_1(t-t_0)};
\tag{10.65}
$$

namely,

$$
\begin{aligned}
\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|&\le K\|\boldsymbol{x}(t_0)-\boldsymbol{y}(t_0)\|\prod_{t_0\le\tau_k<t}(1+KL_1)e^{(\beta+KL_1)(t-t_0)}\\
&=K\|\boldsymbol{x}(t_0)-\boldsymbol{y}(t_0)\|(1+KL_1)^{\mathfrak{N}(t_0,t)}e^{(\beta+KL_1)(t-t_0)}.
\end{aligned}
\tag{10.66}
$$

Let us assume that $\mathfrak{N}(t_0, t) = \varpi(t_0, t)(t - t_0)$ then we have

$$\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\| \leq K\|\boldsymbol{x}(t_0) - \boldsymbol{y}(t_0)\| \times \exp\left[\left(\beta + KL_1 + \frac{\ln(1 + KL_1)}{\varpi(t_0, t)}\right)(t - t_0)\right] \tag{10.67}$$

from which we know that if $\beta + KL_1 + \frac{\ln(1+KL_1)}{\varpi(t_0,t)} < 0$ then solutions of system (10.56) are asymptotically stable. Therefore its almost periodic solution is unique and asymptotically stable. This finishes the proof of the following theorem.

Theorem 10.3.4. *Assume that*

1. *Λ has no eigenvalues on the imaginary axis of the s-plane;*
2. *The following Lipschitz condition holds uniformly with respect to $t \in \mathbb{R}$ and $k \in \mathbb{Z}$:*

$$\|\boldsymbol{f}(t, \boldsymbol{x}_1) - \boldsymbol{f}(t, \boldsymbol{x}_2)\| + \|\boldsymbol{c}_k(\boldsymbol{x}_1) - \boldsymbol{c}_k(\boldsymbol{x}_2)\| \leq K\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{10.68}$$

Then for a sufficiently small K the impulsive control system (10.56) has

1. *a unique almost periodic solution ;*
2. *a unique almost periodic solution that is asymptotically stable if all eigenvalues of Λ are in the left half s-plane.*

⊠

Note 10.3.1. The main results of this chapter are adopted from [3, 27]. Many proofs of theorems had been rewritten to either avoid typos or provide more detailed proofs. If $A(t) \in \mathcal{PC}[(\tau_k, \tau_{k+1}], \mathbb{R}^{n\times n}], k \in \mathbb{Z}$, all results in this chapter still hold. ⧫

11. Applications to Nanoelectronics

In this chapter we study applications of impulsive control theory to nanoelectronics. First, we present models of impulsive electronic devices that are ideal models of nanoelectronic devices. Then, we study some examples of nanoelectronic circuits consisting of driven single-electron tunneling junctions (SETJ) and other nanoelectronic devices.

11.1 Models of Impulsive Electronic Devices

In conventional electronics, there are four basic variables: voltage v, current i, charge q and flux ϕ. There are four kinds of basic electronic devices to model the relations between them[6].

1. A *resistor* is defined by the relationship between current i, voltage v and time t as
$$f_R(v, i, t) = 0;$$
2. An *inductor* is defined by the relationship between flux ϕ, current$i$ and time t as
$$f_L(\phi, i, t) = 0;$$
3. A *capacitor* is defined by the relationship between charge q, voltage v and time t as
$$f_C(q, v, t) = 0;$$
4. A *memristor* is defined by the relationship between charge q, flux ϕ and time t as
$$f_M(q, \phi, t) = 0.$$

So far, there is no systematic research on developing a general framework to handle nanoelectronic devices that containing impulse effects such as electron tunneling. In this section, some ideal models of different kinds of nanoelectronic devices with tunneling effects are present.

Let us introduce the following set of *nanoelectronic devices with impulse effects* (*impulsive devices* for short). Since only the charge q and the flux ϕ can be results of some integral curves, the impulsive effects can only happen to q and ϕ. In contrast of four elementary device models in classical circuit theory,

T. Yang: Impulsive Control Theory, LNCIS 272, pp. 307−333, 2001.

there are only three kinds of elementary impulsive devices in impulsive circuit theory.

Definition 11.1.1. *Given charge q, flux ϕ, time t and moments of tunneling events, then*

1. *An impulsive inductor is defined by*

$$\begin{cases} f_L(\phi, i, t) = 0, & t \neq \tau_k, \\ \Delta\phi = \phi_k, & t = \tau_k; \end{cases} \tag{11.1}$$

2. *An impulsive capacitor is defined by*

$$\begin{cases} f_C(q, i, t) = 0, & t \neq \tau_k, \\ \Delta q = q_k, & t = \tau_k; \end{cases} \tag{11.2}$$

3. *An impulsive memristor is defined by*

$$\begin{cases} f_M(q, \phi, t) = 0, \; t \neq \tau_k, \\ \left.\begin{array}{l} \Delta q = q_k \\ \Delta\phi = \phi_k \end{array}\right\}, \quad t = \tau_k. \end{cases} \tag{11.3}$$

⊠

There are two kinds of impulsive independent sources as defined below.

Definition 11.1.2.

1. *An impulsive current source is given by $q\delta(\tau_k)$ such that a charge q is applied impulsively at moment τ_k to any electronic device to which this impulsive current source is connected in series.*
2. *An impulsive voltage source is given by $\phi\delta(\tau_k)$ such that a flux ϕ is applies impulsively at moment τ_k to any electronic device to which this impulsive current source is connected in parallel.*

⊠

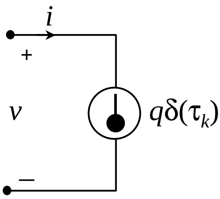
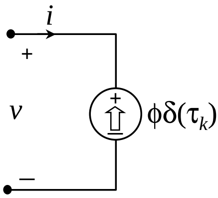
The symbols for impulsive current source and impulsive voltage source are shown in Table 11.1.

We then define four kinds of impulsive dependent sources; namely, impulsive controlled voltage and current sources.

Definition 11.1.3. *Let $v = v(t), t \in \mathbb{R}_+$ be a voltage and $i = i(t), t \in \mathbb{R}_+$ be a current, then*

1. *an impulsive voltage-controlled current source(IVCCS) is given by $q(v)\delta(\tau_k)$ such that a charge $q(v)$ is applied impulsively at moment τ_k to any electronic device to which this IVCCS is connected in series.*
2. *an impulsive current-controlled current source(ICCCS) is given by $q(i)\delta(\tau_k)$ such that a charge $q(i)$ is applied impulsively at moment τ_k to any electronic device to which this ICCCS is connected in series.*

Table 11.1. Symbols of impulsive independent sources.

Type of sources	symbols	characterization
impulsive current source		$\Delta q(\tau_k) = q$
impulsive voltage source		$\Delta \phi(\tau_k) = \phi$

3. *an impulsive current-controlled voltage source(ICCVS) is given by* $\phi(i)\delta(\tau_k)$ *such that a flux* $\phi(i)$ *is applied impulsively at moment* τ_k *to any electronic device to which this ICCVS is connected in parallel.*
4. *an impulsive voltage-controlled voltage source(IVCVS) is given by* $\phi(v)\delta(\tau_k)$ *such that a flux* $\phi(v)$ *is applied impulsively at moment* τ_k *to any electronic device to which this IVCVS is connected in parallel.*

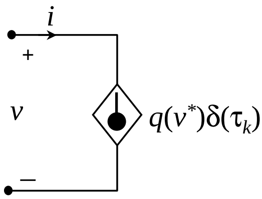
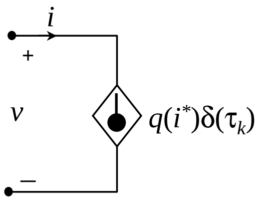
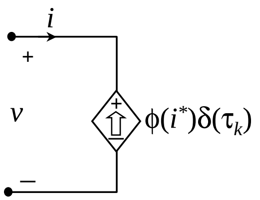
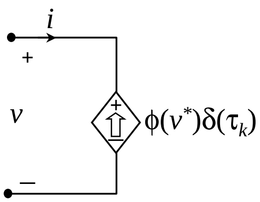
⊠

Table 11.2 shows the symbols for four elementary impulsive controlled sources.

Remark 11.1.1. In fact, to model impulsive electronic circuits , it is enough to add two impulsive independent sources and four impulsive dependent sources into classical electronic circuits. This is because an impulsive inductor can be modeled by a classical inductor connected in parallel with an impulsive voltage source, an impulsive capacitor can be modeled by a classical capacitor connected in series with an impulsive current source, and an impulsive memristor can be modeled by a classical memristor connected in series with with an impulsive current source and in parallel with an impulsive voltage source. However, in many cases, it may be convenient to keep the models of these three kinds of elementary impulsive electronic devices because they can keep the physical integration of electronic devices. ♦

Example 11.1.1. We then present an example of a kind of impulsive capacitor called *single electron tunneling junction*(SETJ) where an electron can tunnel through the barrier in a time period of the order of 10^{-15}s and causes the voltage of the junction capacitor a jump in the order of 10^{-3}V. Since the

Table 11.2. Symbols of impulsive controlled sources.

Type of sources	symbols	characterization
IVCCS		$\Delta q(\tau_k) = q(v^*)$
ICCCS		$\Delta q(\tau_k) = q(i^*)$
ICCVS		$\Delta \phi(\tau_k) = \phi(i^*)$
IVCVS		$\Delta \phi(\tau_k) = \phi(v^*)$

junction capacitor voltage is a state variable of the SETJ model, it is practical to model SETJ using impulsive differential equations. Since SETJ had been used in many nanoelectronic circuit models, the impulsive control theory becomes a very important tool for designing and programming nanodevices which will be the building block for the next generation of computers.

The symbol of a SETJ is shown in Fig. 11.1(a). Observe that a SETJ is modeled by one explicit parameter; namely, the junction capacitance C. The charge, q, in the junction capacitor is the essential variable of the SETJ. If we choose the charge q as the state variable, then the SETJ is modeled by the following impulsive differential equation:

$$\frac{dq}{dt} = i, \quad \text{if } |v(t)| < V_T,$$
$$\Delta q(\tau_k) = \begin{cases} -e, & v(\tau_k) \geq V_T, \\ e, & v_c(\tau_k) \leq -V_T \end{cases} \tag{11.4}$$

where e is the electron charge and

$$V_T = \frac{e}{2C}$$

is the quantum-mechanical *tunneling voltage.*

If we choose the junction voltage v as the state variable, then the SETJ is modeled by the following impulsive differential equation:

$$\frac{dv(t)}{dt} = \frac{1}{C}i, \quad \text{if } |v(t)| < V_T,$$

$$\begin{aligned} \Delta v(\tau_k) &\triangleq v(\tau_k^+) - v(\tau_k) \\ &= \begin{cases} -2V_T, & v(\tau_k) \geq V_T, \\ 2V_T, & v(\tau_k) \leq -V_T. \end{cases} \end{aligned} \tag{11.5}$$

It is easy to see that the SETJ is equivalent to the impulsive electronic circuit shown in Fig. 11.1(b) where the junction capacitance of the SETJ is assigned to a linear capacitor and the tunneling effects of the SETJ is modeled by an IVCCS.

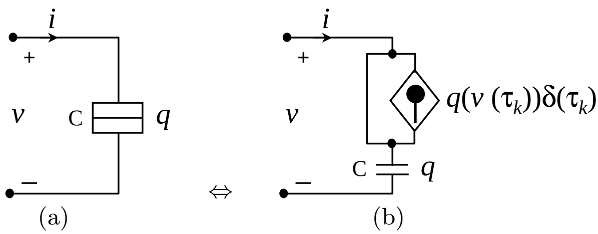


Fig. 11.1. (a) The symbol of the SETJ. (b) The equivalent impulsive electronic circuit of the SETJ.

★

11.2 Driven SETJ Electronic Circuit

The block diagram of an isolated SETJ circuit biased by a DC voltage source V_b and driven by a sinusoidal pump $v_p(t)$ is shown in Fig. 11.2. In this section we call this impulsive electronic circuit as a driven SETJ electronic circuit. This impulsive electronic circuit has been investigated for implementing logic operations via "tunnel phase logic"(TPL) in [23, 18, 12, 47, 45]. This TPL operation has many potential applications in digital technology[23, 18, 12]. In a TPL operation, the phase of periodic solutions are used to encode digital operations, therefore, the periodic solutions of driven SETJ circuits are of great interest.

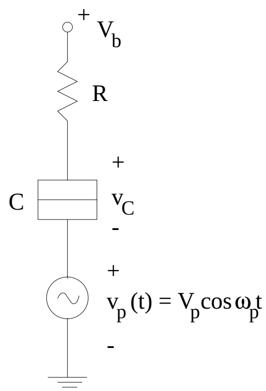


Fig. 11.2. A driven single-electron tunneling junction circuit.

The driven SETJ circuit in Fig. 11.2 can be modeled by a nonautonomous impulsive differential equation. Let us choose the state variable as the voltage, v_C, across the SETJ. The sinusoidal voltage source (pump) is defined by

$$v_p(t) = V_p \cos(\omega_p t).$$

The DC voltage source V_b is in series with a resistor R.

Remark 11.2.1. Observe that in a driven SETJ circuit, the plant itself is impulsive and we can use the AC pump signal and the DC bias as continuous control inputs. Therefore, the problem of control a driven SETJ circuit to a T-periodic solution is a type-III impulsive control problem . ♦

11.2.1 Circuit Model and its Dimensionless Form

The dynamics of the SETJ circuit in Fig. 11.2 is governed by the following nonautonomous impulsive differential equation:

$$\begin{aligned} \frac{dv_C(t)}{dt} &= \frac{1}{RC}(V_b - v_C(t) - V_p\cos(\omega_p t)), \quad \text{if } |v_C(t)| < V_T, \\ \Delta v_C(t_i) &\triangleq v_c(t_i^+) - v_c(t_i) \\ &= \begin{cases} -2V_T, & v_c(t_i) \geq V_T, \\ 2V_T, & v_c(t_i) \leq -V_T, \end{cases} \end{aligned} \tag{11.6}$$

where $\{t_1, t_2, \cdots, t_i, \cdots\}$ are moments when $v_C(t_i) = \pm V_T$. Here $v_C(t)$ is the junction voltage, and

$$V_T = \frac{e}{2C}.$$

By simple algebra we can normalize (11.6) into the following dimensionless form:

$$\begin{aligned}
\frac{d\theta}{d\tau} &= \gamma\left(a - b\cos\tau - \frac{\theta}{\pi}\right), \qquad \text{if } |\theta(t)| < \pi, \\
\Delta\theta(\tau_i) &\triangleq \theta(\tau_i^+) - \theta(\tau_i) \\
&= \begin{cases} -2\pi, & \theta(\tau_i) \geq \pi, \\ 2\pi, & \theta(\tau_i) \leq -\pi, \end{cases}
\end{aligned} \tag{11.7}$$

where

$$a = \frac{V_b}{V_T}, \quad b = \frac{V_p}{V_T}, \quad \theta = \frac{2\pi C v_C}{e}, \quad \tau = \omega_p t, \quad \gamma = \frac{\pi}{RC\omega_p}.$$

Without loss of generality (11.7) can be simplified as:

$$\begin{aligned}
\frac{d\theta}{d\tau} &= \gamma\left(a - b\cos\tau - \frac{\theta}{\pi}\right), \qquad \text{if } \theta(t) < \pi, \\
\Delta\theta(\tau_i) &= -2\pi, \qquad \text{if } \theta(\tau_i) = \pi, \\
&|\theta(0)| < \pi
\end{aligned} \tag{11.8}$$

where

$$\{\tau_i\} \triangleq \{\tau_i |\ \theta(\tau_i) = \pi\}. \tag{11.9}$$

For convenience, we use t to denote the normalized time τ in (11.8) and rewrite (11.8) into the following more compact form of nonautonomous impulsive differential equations:

$$\begin{aligned}
\frac{d\theta}{dt} &= -\frac{\gamma}{\pi}\theta + \gamma(a - b\cos t), \qquad t \neq \tau_i, \\
\Delta\theta(\tau_i) &= -2\pi, \qquad t = \tau_i, \\
|\theta(0)| &< \pi.
\end{aligned} \tag{11.10}$$

where $\{\tau_1, \tau_2, \cdots, \tau_i, \cdots\}$ are moments when $\theta(t) = \pi$.

11.2.2 *T*-periodic Solutions

In this section we give the conditions under which a driven SETJ circuit has T-periodic solutions, where T is a positive real number. Since the pump source $b\cos(t)$ is 2π-periodic and the tunneling events for a T-periodic solution should also be T-periodic, we have

$$T = 2n\pi, \qquad \tau_{i+p} = \tau_i + T, \tag{11.11}$$

where n and p are positive integers. The following theorem gives the analytical form of the T-periodic solutions of the driven SETJ circuit.

Theorem 11.2.1. *There exists a $T = 2n\pi$ (n is some positive integer) such that the dimensionless SETJ equation (11.10) has n distinct T-periodic solutions given by*

$$\begin{aligned}\theta^*(t) = {} & e^{-\frac{\gamma}{\pi}t}\frac{1}{1-e^{-\frac{\gamma}{\pi}T}}\left[\int_0^T e^{-\frac{\gamma}{\pi}(T-\tau)}\gamma(a - b\cos(\tau + 2q\pi))d\tau \right.\\ & \left. - \sum_{i=1}^{p} 2\pi e^{-\frac{\gamma}{\pi}(T-\tau_i)}\right]\\ & + \int_0^t e^{-\frac{\gamma}{\pi}(t-\tau)}\gamma(a - b\cos(\tau + 2q\pi))d\tau - \sum_{0<\tau_i<t} 2\pi e^{-\frac{\gamma}{\pi}(t-\tau_i)},\\ & q = 0, 1, \cdots, n-1.\end{aligned} \tag{11.12}$$

where $\tau_{i+p} = \tau_i + T$. ⊠

Proof. Let the initial condition be $\theta(0, \theta_0) = \theta_0$. Any solution $\theta(t, \theta_0)$ of (11.10) is given by

$$\begin{aligned}\theta(t, \theta_0) = {} & e^{-\frac{\gamma}{\pi}t}\theta_0 + \int_0^t e^{-\frac{\gamma}{\pi}(t-\tau)}\gamma(a - b\cos\tau)d\tau \\ & - \sum_{0<\tau_i<t} 2\pi e^{-\frac{\gamma}{\pi}(t-\tau_i)}.\end{aligned} \tag{11.13}$$

Since $\gamma(a - b\cos\tau)$ is a 2π-periodic function, a 2π phase shift will result in a new solution sharing the same initial condition θ_0. Thus, (11.13) can be rewritten as

$$\begin{aligned}\theta(t, \theta_0) = {} & e^{-\frac{\gamma}{\pi}t}\theta_0 + \int_0^t e^{-\frac{\gamma}{\pi}(t-\tau)}\gamma(a - b\cos(\tau + 2q\pi))d\tau \\ & - \sum_{0<\tau_i<t} 2\pi e^{-\frac{\gamma}{\pi}(t-\tau_i)}, \qquad q = 0, 1, 2, \cdots.\end{aligned} \tag{11.14}$$

For T-periodic solutions that has θ_0 as its initial state, we have the following condition:

$$\begin{aligned}(1 - e^{-\frac{\gamma}{\pi}T})\theta_0 = {} & \int_0^T e^{-\frac{\gamma}{\pi}(T-\tau)}\gamma(a - b\cos(\tau + 2q\pi))d\tau \\ & - \sum_{i=1}^{p} 2\pi e^{-\frac{\gamma}{\pi}(T-\tau_i)}, \qquad q = 0, 1, 2, \cdots, n-1.\end{aligned} \tag{11.15}$$

Since $(1 - e^{-\frac{\gamma}{\pi}T}) \neq 0$, it follows that

$$\theta_0 = \frac{1}{1 - e^{-\frac{\gamma}{\pi}T}} \left[\int_0^T e^{-\frac{\gamma}{\pi}(T-\tau)} \gamma(a - b\cos(\tau + 2q\pi)) d\tau \right.$$
$$\left. - \sum_{i=1}^{p} 2\pi e^{-\frac{\gamma}{\pi}(T-\tau_i)} \right], \qquad q = 0, 1, 2, \cdots, n-1. \tag{11.16}$$

Substituting (11.16) into (11.13) we obtain the following n distinct T-periodic solutions:

$$\theta^*(t) = e^{-\frac{\gamma}{\pi}t} \frac{1}{1 - e^{-\frac{\gamma}{\pi}T}} \left[\int_0^T e^{-\frac{\gamma}{\pi}(T-\tau)} \gamma(a - b\cos(\tau + 2q\pi)) d\tau \right.$$
$$\left. - \sum_{i=1}^{p} 2\pi e^{-\frac{\gamma}{\pi}(T-\tau_i)} \right] + \int_0^t e^{-\frac{\gamma}{\pi}(t-\tau)} \gamma(a - b\cos(\tau + 2q\pi)) d\tau$$
$$- \sum_{0<\tau_i<t} 2\pi e^{-\frac{\gamma}{\pi}(t-\tau_i)}, \qquad q = 0, 1, 2, \cdots, n-1. \tag{11.17}$$

■

Remark 11.2.2. Observe that Theorem 11.2.1 asserts that the SETJ circuit in Fig. 11.2 has at least one T-periodic solution. So far, a $T = 4n\pi$ solution has been found in [23]. This theorem shows that for the same periodic orbit (limit cycle) associated with a driven SETJ, the phases of the tunneling events (i.e., the moments of impulse effects) can be shifted by 2π without changing the initial condition which gives rise to a T-periodic solution. It also shows that a driven SETJ can have n distinct $2n\pi$-periodic solutions. Thus, we immediately have the following corollary. ♦

Corollary 11.2.1. *If the SETJ equation (11.10) has a $2n\pi$-periodic solution, then there are $(n-1)$ additional distinct $2n\pi$-periodic solutions with 2π phase shifts between consecutive solution pairs.* ⊠

Remark 11.2.3. The existence of n distinct $2n\pi$-periodic solutions in a driven SETJ circuit can be used to build robust digital memories as suggested in [23]. This is because the different phase shifts can be used to represent different memory states. For example, to build a binary digital memory, we must choose SETJs that have 4π-periodic solutions. The coexistence of two 4π-periodic solutions has been reported in [23]. ♦

11.3 Return Maps of Driven SETJ Circuits

Since the driven SETJ circuit in (11.10) is driven by a periodic signal, it is convenient to construct return maps based on the period of the pump signal. The *pth-time return map* can be defined by $\Pi_p : \theta_n \mapsto \theta_{n+p}$, where $\theta_n = \theta(2n\pi)$. In [23], the authors presented some numerical results on Π_2. In this section, we will present both theoretical and numerical results on Π_p, $p = 1, 2, \cdots$. In this section, we will use extensively the following formula:

$$\int e^{ax}\cos(bx)dx = \frac{1}{a^2+b^2}e^{ax}[a\cos(bx) + b\sin(bx)] + c \tag{11.18}$$

where c is a constant.

To study the return map $\Pi_1 : \theta_n \mapsto \theta_{n+1}$, we can choose $\theta_n \in [-\pi, \pi)$ as initial conditions at $t = 0$ and calculate θ_{n+1} at time $t = 2\pi$. Using (11.13) with initial condition $\theta_0 = \theta_n$, Π_1 is given by the solution of the following two equations:

$$\begin{aligned}
\theta_{n+1} &= e^{-\frac{\gamma}{\pi}(2\pi)}\theta_n + \int_0^{2\pi} e^{-\frac{\gamma}{\pi}(2\pi-\tau)}\gamma(a - b\cos\tau)d\tau \\
&\quad - \sum_{0<\tau_i<2\pi} 2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_i)} \\
&= e^{-2\gamma}\theta_n + a\pi - a\pi e^{-2\gamma} - b\gamma e^{-2\gamma}\int_0^{2\pi} e^{\frac{\gamma}{\pi}\tau}\cos\tau d\tau \\
&\quad - \sum_{0<\tau_i(\theta_n)<2\pi} 2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_i(\theta_n))} \\
&= e^{-2\gamma}\theta_n + (1 - e^{-2\gamma})\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right) \\
&\quad - \sum_{0<\tau_i(\theta_n)<2\pi} 2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_i(\theta_n))}, \\
\theta(\tau_i(\theta_n), \theta_n) &= \pi, \qquad \tau_i(\theta_n) \in (0, 2\pi).
\end{aligned} \tag{11.19}$$

Observe that the only uncertainty comes from impulsive events. Let us consider first the case when there is no impulsive event for a given θ_n:

$$\begin{aligned}
\theta(t, \theta_n) &= e^{-\frac{\gamma}{\pi}t}\theta_n + \int_0^t e^{-\frac{\gamma}{\pi}(t-\tau)}\gamma(a - b\cos\tau)d\tau \\
&= e^{-\frac{\gamma}{\pi}t}\theta_n + a\pi(1 - e^{-\frac{\gamma}{\pi}t}) \\
&\quad - \frac{b\gamma\pi^2}{\gamma^2+\pi^2}\left(\sin(t) + \frac{\gamma}{\pi}\cos(t) - \frac{\gamma}{\pi}e^{-\frac{\gamma}{\pi}t}\right) \\
&< \pi, \qquad \text{for a given } \theta_n \in [-\pi, \pi) \text{ and } t \in (0, 2\pi].
\end{aligned} \tag{11.20}$$

By combining the results in (11.19) and (11.20) we can derive a closed form expression for Π_1 when there is no impulsive event as follows:

$$\begin{aligned}
\theta_{n+1} &= e^{-2\gamma}\theta_n + (1-e^{-2\gamma})\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right), \\
\theta(t,\theta_n) &= e^{-\frac{\gamma}{\pi}t}\theta_n + a\pi(1-e^{-\frac{\gamma}{\pi}t}) \\
&\quad -\frac{b\gamma\pi^2}{\gamma^2+\pi^2}\left(\sin(t) + \frac{\gamma}{\pi}\cos(t) - \frac{\gamma}{\pi}e^{-\frac{\gamma}{\pi}t}\right) \\
&< \pi, \qquad \text{for a given } \theta_n \in [-\pi,\pi) \text{ and } t \in (0,2\pi]. \qquad (11.21)
\end{aligned}$$

This result shows that if there is no impulsive event in a driven SETJ circuit, then Π_1 is a straight line with a slope $e^{-2\gamma}$ and a shift given by

$$\alpha \triangleq (1-e^{-2\gamma})\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right)$$

along the θ_{n+1}-axis. Observe that the slope $e^{-2\gamma} < 1$ is always satisfied because of physical settings. Hence, if $\alpha \in [-\pi,\pi)$, we can guarantee that the 2π-periodic solutions are asymptotically stable.

Let us consider next the situation when there is an impulsive event within any 2π time period. The interest is to find the initial condition θ^* for a 2π-periodic solution which has only one impulsive event within each period. In this case, $\theta_n = \theta_{n+1} \triangleq \theta^*$ should hold and if $\tau_1^* \in (0,2\pi]$ is the moment of the impulsive event, we can write

$$\theta^* = e^{-2\gamma}\theta^* + \left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right)(1-e^{-2\gamma}) - 2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_1^*)}. \qquad (11.22)$$

In this equation, there are two unknown parameters θ^* and τ_1^*. We need to construct one more equation in order to find both unknown parameters. One condition we know is that $\theta(\tau_1^*,\theta^*) = \pi$ which leads to the following equation:

$$\begin{aligned}
\pi &= e^{-\frac{\gamma}{\pi}\tau_1^*}\theta^* + a\pi(1-e^{-\frac{\gamma}{\pi}\tau_1^*}) \\
&\quad -\frac{b\gamma\pi^2}{\gamma^2+\pi^2}\left(\sin(\tau_1^*) + \frac{\gamma}{\pi}\cos(\tau_1^*) - \frac{\gamma}{\pi}e^{-\frac{\gamma}{\pi}\tau_1^*}\right). \qquad (11.23)
\end{aligned}$$

Since $1 - e^{-2\gamma} \neq 0$ is guaranteed by physical settings, from (11.22) we have

$$\begin{aligned}
\theta^* &= \frac{1}{1-e^{-2\gamma}}\left[\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right)(1-e^{-2\gamma}) - 2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_1^*)}\right] \\
&= a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2} - \frac{2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_1^*)}}{1-e^{-2\gamma}}. \qquad (11.24)
\end{aligned}$$

Substituting (11.24) into (11.23) we have

$$\begin{aligned}
\pi &= e^{-\frac{\gamma}{\pi}\tau_1^*}\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2} - \frac{2\pi e^{-\frac{\gamma}{\pi}(2\pi-\tau_1^*)}}{1-e^{-2\gamma}}\right) + a\pi(1-e^{-\frac{\gamma}{\pi}\tau_1^*}) \\
&\quad -\frac{b\gamma\pi^2}{\gamma^2+\pi^2}\left(\sin(\tau_1^*) + \frac{\gamma}{\pi}\cos(\tau_1^*) - \frac{\gamma}{\pi}e^{-\frac{\gamma}{\pi}\tau_1^*}\right). \qquad (11.25)
\end{aligned}$$

The solutions of this equation gives τ_1^*. We can solve this equation by numerical methods. The reasonable solution of τ_1^* should be in interval $(0, 2\pi]$. If there is no solution in interval $(0, 2\pi]$, then it means that there is not 2π-periodic solution with this set of parameters.

As for other initial condition $\theta_n \neq \theta^*$, the moment of the impulsive event satisfies $\tau_1 \neq \tau_1^*$. The return map is then given by the solution of the following equations:

$$\begin{aligned}
\theta_{n+1} &= e^{-2\gamma}\theta_n + (1 - e^{-2\gamma})\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right) \\
&\quad -2\pi e^{-\frac{\gamma}{\pi}(2\pi - \tau_1(\theta_n))}, \\
\theta(\tau_1(\theta_n), \theta_n) &= e^{-\frac{\gamma}{\pi}\tau_1(\theta_n)}\theta_n + a\pi(1 - e^{-\frac{\gamma}{\pi}\tau_1(\theta_n)}) \\
&\quad -\frac{b\gamma\pi^2}{\gamma^2+\pi^2}\left(\sin(\tau_1(\theta_n)) + \frac{\gamma}{\pi}\cos(\tau_1(\theta_n)) - \frac{\gamma}{\pi}e^{-\frac{\gamma}{\pi}\tau_1(\theta_n)}\right) \\
&= \pi, \qquad \tau_1(\theta_n) \in (0, 2\pi).
\end{aligned} \tag{11.26}$$

Remark 11.3.1. Using the same method presented in this section we can derive many other return maps. Observe that whenever an impulsive event occurs in a return map, there will be an exponential effect on the straight line segments in the return map. Π_p is the solution of the following equations:

$$\begin{aligned}
\theta_{n+p} &= e^{-\frac{\gamma}{\pi}(2p\pi)}\theta_n + \int_0^{2p\pi} e^{-\frac{\gamma}{\pi}(2p\pi-\tau)}\gamma(a - b\cos\tau)d\tau \\
&\quad - \sum_{0<\tau_i<2p\pi} 2\pi e^{-\frac{\gamma}{\pi}(2p\pi - \tau_i)} \\
&= e^{-2p\gamma}\theta_n + a\pi - a\pi e^{-2p\gamma} - b\gamma e^{-2p\gamma}\int_0^{2p\pi} e^{\frac{\gamma}{\pi}\tau}\cos\tau d\tau \\
&\quad - \sum_{0<\tau_i(\theta_n)<2p\pi} 2\pi e^{-\frac{\gamma}{\pi}(2p\pi - \tau_i(\theta_n))} \\
&= e^{-2p\gamma}\theta_n + (1 - e^{-2p\gamma})\left(a\pi - \frac{b\gamma^2\pi}{\pi^2+\gamma^2}\right) \\
&\quad - \sum_{0<\tau_i(\theta_n)<2p\pi} 2\pi e^{-\frac{\gamma}{\pi}(2p\pi - \tau_i(\theta_n))}, \\
\theta(\tau_i(\theta_n), \theta_n) &= \pi, \qquad \tau_i(\theta_n) \in (0, 2p\pi).
\end{aligned} \tag{11.27}$$

The behaviors and theoretical analysis of two-coupled driven SETJ circuits can be found in [44]. ♦

11.4 *T*-Periodic Solutions of a Second-order SETJ Circuit

Let us study the existence of T-periodic solutions of the second-order SETJ circuit shown in Fig. 11.3. In this circuit C_1 is a linear capacitor, C_2 represents the junction capacitance of the SETJ. The state variables are v_1 and v_2 that are voltages across C_1 and C_2, respectively.

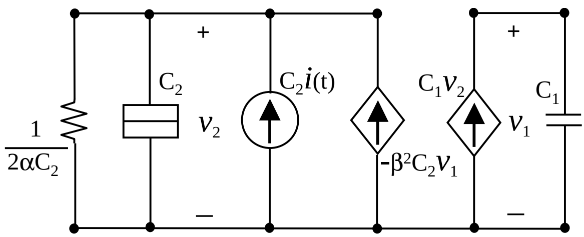


Fig. 11.3. The circuit block diagram of a second-order SETJ circuit.

Let $\{\tau_k(v_2)\}$, $k \in \mathbb{N}$, be the set of time moments when tunneling events happen in the SETJ. When $t \neq \tau_k(v_2)$ we have

$$\begin{aligned} C_1\frac{dv_1}{dt} &= C_1 v_2, \\ C_2\frac{dv_2}{dt} &= -\beta^2 C_2 v_1 - 2\alpha C_2 v_2 + C_2 i(t). \end{aligned} \tag{11.28}$$

Let $q_1(t)$ and $q_2(t)$ be the charges in C_1 and C_2, respectively, when $t = \tau_k(v_2)$ we have

$$\begin{aligned} &\Delta q_1 = 0, \quad \Rightarrow \quad \Delta v_1 = 0, \\ &\Delta q_2 = \begin{cases} -e, \text{ if } v_2(\tau_k) \geq \tilde{V}_T, \\ e \quad \text{if } v_2(\tau_k) \leq -\tilde{V}_T, \end{cases} \quad \Rightarrow \\ &\Delta v_2 = e/C_2 = c_k \triangleq \begin{cases} -2\tilde{V}_T, \text{ if } v_2(\tau_k) \geq \tilde{V}_T, \\ 2\tilde{V}_T \quad \text{if } v_2(\tau_k) \leq -\tilde{V}_T, \end{cases} \end{aligned} \tag{11.29}$$

where e is the electron charge and

$$\tilde{V}_T = \frac{e}{2C_2}.$$

The state equation of this circuit is then given by

$$\underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{\boldsymbol{v}}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\alpha \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\boldsymbol{v}} + \underbrace{\begin{pmatrix} 0 \\ i(t) \end{pmatrix}}_{\boldsymbol{u}(t)}, \quad t \neq \tau_k(v_2)$$

$$\Delta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ c_k \end{pmatrix}}_{\boldsymbol{c}_k}, \quad t = \tau_k(v_2). \tag{11.30}$$

The eigenvalue $\lambda_{1,2}$ of A are the roots of

$$\lambda^2 + 2\alpha\lambda + \beta^2 = 0. \tag{11.31}$$

Let $\Phi(t) = e^{At}$, which satisfies

$$\dot{\Phi}(t) = A\Phi(t), \tag{11.32}$$

be the normalized fundamental matrix of system (11.30) at $t = 0$, then we have

1. If $\alpha^2 - \beta^2 > 0$, then $\lambda_{1,2}$ are distinct and real and

$$\Phi(t) = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} & e^{\lambda_1 t} - e^{\lambda_2 t} \\ -\lambda_1\lambda_2(e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} \end{pmatrix}; \tag{11.33}$$

2. If $\alpha^2 - \beta^2 = 0$, then $\lambda_{1,2} = -\alpha$ and

$$\Phi(t) = \begin{pmatrix} (1 + \alpha t)e^{-\alpha t} & te^{-\alpha t} \\ -\alpha^2 te^{-\alpha t} & (1 - \alpha t)e^{-\alpha t} \end{pmatrix}; \tag{11.34}$$

3. If $\omega^2 \triangleq \beta^2 - \alpha^2 > 0$, then $\lambda_{1,2} = -\alpha \pm i\omega$ and

$$\Phi(t) = \frac{1}{\omega} \begin{pmatrix} e^{-\alpha t}[\omega \cos(\omega t) + \alpha \sin(\omega t)] & e^{-\alpha t} \sin(\omega t) \\ -(\omega^2 + \alpha^2)e^{-\alpha t} \sin(\omega t) & e^{-\alpha t}[\omega \cos(\omega t) - \alpha \sin(\omega t)] \end{pmatrix}. \tag{11.35}$$

Let us denote the initial voltage of capacitor C_1 and that of the SETJ junction capacitor C_2 as $\boldsymbol{v}(t_0^+) = \boldsymbol{v}_0 \triangleq (v_1(0), v_2(0))^\top$, then for $t > \tau_0$ the solutions of system (11.30) are given by

$$\boldsymbol{v}(t) = \Phi(t - \tau_0)\boldsymbol{v}_0 + \int_{\tau_0}^{t} \Phi(t - s)\boldsymbol{u}(s)ds + \sum_{\tau_0 < \tau_k < t} \Phi(t - \tau_k)\boldsymbol{c}_k. \tag{11.36}$$

Let us assume that there is a $T > 0$ and a $\varrho \in \mathbb{N}$ such that

$$\tau_{k+\varrho}(v_2) = \tau_k(v_2), \; \boldsymbol{u}(t + T) = \boldsymbol{u}(t), \; \boldsymbol{c}_{k+\varrho} = \boldsymbol{c}_k, \; t \in \mathbb{R}, \; k \in \mathbb{Z} \tag{11.37}$$

then the system (11.30) is T-periodic. Let us study the existence of T-periodic solutions of system (11.30). Let $\boldsymbol{v}(\tau_0^+)$ be the initial condition of a T-periodic solution, then we have

$$[I - \Phi(T)]\boldsymbol{v}(\tau_0^+) = \int_{\tau_0}^{\tau_\varrho} \Phi(\tau_\varrho - s)\boldsymbol{u}(s)ds + \sum_{k=1}^{\varrho} \Phi(\tau_\varrho - \tau_k)\boldsymbol{c}_k \quad (11.38)$$

Then we have the following possible situations.

1. If $\lambda_1 \neq i2p\pi/T$ and $\lambda_2 \neq i2q\pi/T$ for all $p, q \in \mathbb{Z}$, then the multipliers $\mu_{1,2} = e^{\lambda_{1,2}T}$ of (11.32) are not equal to 1. In this case, system (11.30) has a unique T-periodic solution whose initial value $\boldsymbol{v}_0$ is given by (11.38).
2. $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case, (11.32) has one linearly independent T-periodic solution
$$\Phi_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
Therefore the adjoint equation to (11.32),
$$\dot{\Psi}(t) = -A^*\Psi(t) \quad (11.39)$$
also has one linearly independent T-periodic solution
$$\Psi_1(t) = \begin{pmatrix} 1 \\ -1/\lambda_1 \end{pmatrix}.$$
Then from (9.68) we have
$$-\int_{\tau_0}^{\tau_\varrho} \frac{i(s)}{\lambda_1}ds - \sum_{k=1}^{\varrho} \frac{c_k}{\lambda_1} = 0. \quad (11.40)$$
If the condition in (11.40) holds, it follows from (11.38) that we can find $\boldsymbol{v}(\tau_0^+) \triangleq (v_1(\tau_0^+), v_2(\tau_0^+))^\top$ by using the following equation
$$\begin{aligned} & v_1(\tau_0^+) \text{ can be arbitrary,} \\ (1 - e^{\lambda_1 T})v_2(\tau_0^+) = & \int_{\tau_0}^{\tau_\varrho} e^{\lambda_1(\tau_\varrho - s)}i(s)ds \\ & + \sum_{k=1}^{\varrho} e^{\lambda_1(\tau_\varrho - \tau_k)}c_k. \end{aligned} \quad (11.41)$$
Thus, system (11.30) has a family of T-periodic solutions.
3. $\lambda_1 = \lambda_2 = 0$. In this case, (11.32) and the adjoint equation (11.39) each has, respectively, one linearly independent T-periodic solution
$$\Phi_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \Psi_1(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then from (9.68) we have

$$\int_{\tau_0}^{\tau_\varrho} i(s)ds + \sum_{k=1}^{\varrho} c_k = 0. \tag{11.42}$$

If the condition in (11.42) holds, it follows from (11.38) that we can find $\boldsymbol{v}(\tau_0^+) = (v_1(\tau_0^+), v_2(\tau_0^+))^\top$ by using the following equation

$$\begin{aligned} &v_1(\tau_0^+) \text{ can be arbitrary,} \\ &v_2(\tau_0^+)T = \int_{\tau_0}^{\tau_\varrho} i(s)sds + \sum_{k=1}^{\varrho} \tau_k c_k. \end{aligned} \tag{11.43}$$

Thus, system (11.30) has a family of T-periodic solutions.

4. $\lambda_{1,2} = \pm i\omega \triangleq i2p\pi/T$ for some $p \in \mathbb{Z}$. In this case, (11.32) and the adjoint equation (11.39) each has, respectively, two linearly independent T-periodic solutions

$$\Phi(t) = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

and

$$\Psi(t) = \begin{pmatrix} \cos(\omega t) & \omega \sin(\omega t) \\ -\frac{1}{\omega}\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Then from (9.68) we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_\varrho} \cos(\omega s) i(s)ds + \sum_{k=1}^{\varrho} \cos(\omega \tau_k) c_k = 0, \\ &\int_{\tau_0}^{\tau_\varrho} \sin(\omega s) i(s)ds + \sum_{k=1}^{\varrho} \sin(\omega \tau_k) c_k = 0. \end{aligned} \tag{11.44}$$

If the condition in (11.44) holds, then any $\boldsymbol{v}(\tau_0^+) = (v_1(\tau_0^+), v_2(\tau_0^+))^\top$ satisfies (11.38). Therefore, all solutions of system (11.30) are T-periodic.

Example 11.4.1. In this example let us choose parameters as

$$\alpha = 2, \quad \beta = 1, \quad T = 1, \text{ and } c_k = \begin{cases} -2, & k \text{ is odd,} \\ 2, & k \text{ is even.} \end{cases}$$

We have $\lambda_{1,2} = -2 \pm \sqrt{3} < 0$, therefore the T-periodic solutions are stable. In the simulations shown in Fig. 11.4 we choose the AC pump current source as

$$i(t) = 10\sin(2\pi t).$$

From Fig. 11.4(a) we can see that the system approaches a limit cycle asymptotically. From Fig. 11.4(b) we can see that the solution, which the system approaches, is a T-period solution. However, within each period, there are two tunneling events.

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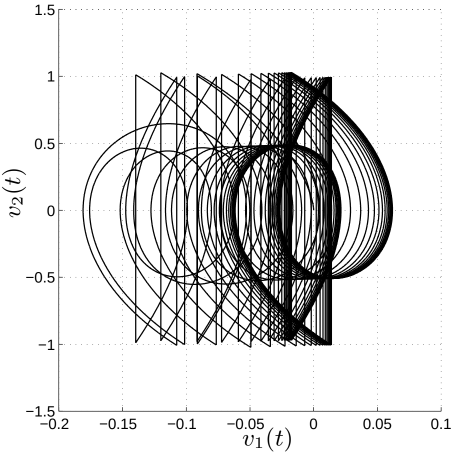
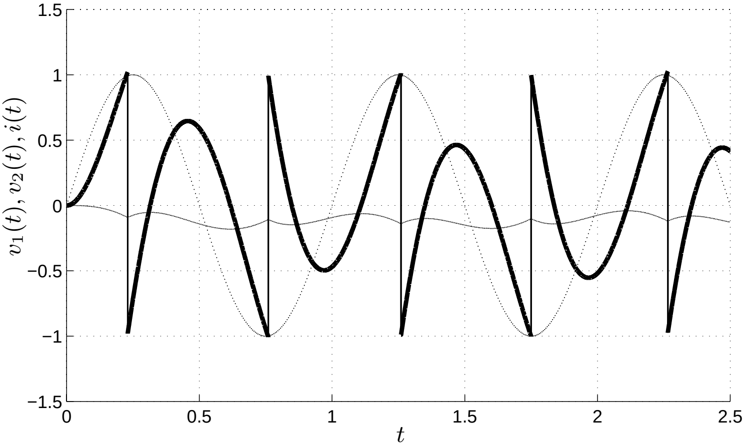


(a)



(b)

Fig. 11.4. The simulation results of controlling nanoelectronic systems to T-periodic solutions. (a) $v_1(t)$ versus $v_2(t)$ plot. (b) Waveforms of $v_1(t)$(thin-solid), $v_2(t)$(thick-solid) and $i(t)$(dotted).

11.5 *T*-Periodic Solutions of a Nanoelectronic Circuit Consisting of IVCCS

Let us study the existence of T-periodic solutions of the second-order SETJ circuit shown in Fig. 11.5. In this circuit, there is an impulsive voltage control current source (IVCCS) connected in series with the SETJ. C_1 is a linear capacitor and C_2 is the junction capacitor of the SETJ. Observe that the IVCCS is triggered by the tunneling events of the SETJ.

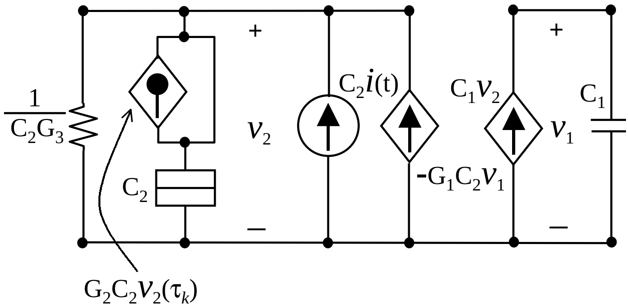


Fig. 11.5. The circuit block diagram of a second-order SETJ circuit with impulsive voltage control current source controlled by tunneling events of the SETJ. $\tau_k, k \in \mathbb{N}$ are time moments of tunneling events in the SETJ.

Let $\tau_k(v_2), k \in \mathbb{N}$ be time moments of tunneling events in the SETJ, when $t \neq \tau_k(v_2)$ we have

$$\begin{aligned} C_1\frac{dv_1}{dt} &= C_1 v_2, \\ C_2\frac{dv_2}{dt} &= -C_2G_1v_1 - C_2G_3v_2 + C_2 i(t), \end{aligned} \tag{11.45}$$

where $G_1 > 0$, $G_3 > 0$, $\tau_k = kT, k \in \mathbb{Z}$. $i(t)$ is a T-periodic current source and we assume that $G_3^2 \neq 4G_1$.

Let $q_1(t)$ and $q_2(t)$ be charges in C_1 and C_2, respectively. When $t = \tau_k(v_2)$ we have

$$\begin{aligned} \Delta q_1 &= 0, \qquad\qquad\qquad\qquad \Rightarrow \Delta v_1 = 0, \\ \Delta q_2 &= C_2 \Delta v_2 \\ &= Q_k + \int_{-\infty}^{\infty} C_2G_2v_2(\tau_k)\delta(\tau_k)dt = Q_k + C_2G_2v_2(\tau_k), \qquad \Rightarrow \\ \Delta v_2 &= G_2 v_2 + c_k, \end{aligned} \tag{11.46}$$

where

$$Q_k = \begin{cases} -e, & \text{if } v_2(\tau_k) \geq \tilde{V}_T, \\ e, & \text{if } v_2(\tau_k) \leq -\tilde{V}_T, \end{cases} \tag{11.47}$$

$$c_k = \begin{cases} -2\tilde{V}_T, & \text{if } v_2(\tau_k) \geq \tilde{V}_T, \\ 2\tilde{V}_T, & \text{if } v_2(\tau_k) \leq -\tilde{V}_T, \end{cases} \tag{11.48}$$

and

$$\tilde{V}_T = \frac{e}{2C_2}.$$

The state equation of this circuit is given by

$$\begin{aligned}
\underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{\boldsymbol{v}}} &= \underbrace{\begin{pmatrix} 0 & 1 \\ -G_1 & -G_3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\boldsymbol{v}} + \underbrace{\begin{pmatrix} 0 \\ i(t) \end{pmatrix}}_{\boldsymbol{u}(t)}, \quad t \neq \tau_k(v_2) \\
\Delta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}}_{B} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ c_k \end{pmatrix}}_{\boldsymbol{c}_k}, \quad t = \tau_k(v_2),
\end{aligned} \tag{11.49}$$

The companying homogeneous system of (11.49) is given by

$$\begin{aligned}
\underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{\boldsymbol{v}}} &= \underbrace{\begin{pmatrix} 0 & 1 \\ -G_1 & -G_3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\boldsymbol{v}}, \quad t \neq \tau_k(v_2) \\
\Delta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}}_{B} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad t = \tau_k(v_2),
\end{aligned} \tag{11.50}$$

Let the $\lambda_{1,2}$ be the eigenvalues of A, then we have

$$\lambda_{1,2} = \frac{-G_3 \pm \sqrt{G_3^2 - 4G_1}}{2} \tag{11.51}$$

and

$$e^{At} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} & e^{\lambda_1 t} - e^{\lambda_2 t} \\ -\lambda_1 \lambda_2 (e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} \end{pmatrix}. \tag{11.52}$$

The monodromy matrix for (11.50) is given by

$$\begin{aligned}
M = (I + B)e^{AT} &= \frac{1}{\lambda_1 - \lambda_2} \\
&\times \begin{pmatrix} \lambda_1 e^{\lambda_2 T} - \lambda_2 e^{\lambda_1 T} & e^{\lambda_1 T} - e^{\lambda_2 T} \\ -(1+G_2)\lambda_1 \lambda_2 (e^{\lambda_1 T} - e^{\lambda_2 T}) & (1+G_2)(\lambda_1 e^{\lambda_1 T} - \lambda_2 e^{\lambda_2 T}) \end{pmatrix},
\end{aligned} \tag{11.53}$$

whose eigenvalues are the multipliers, $\mu_{1,2}$, given by the roots of the following equation:

$$\begin{aligned}
&\mu^2 - \frac{1}{\lambda_1 - \lambda_2} \left\{ \lambda_1 e^{\lambda_2 T} - \lambda_2 e^{\lambda_1 T} + (1+G_2)\left[\lambda_1 e^{\lambda_1 T} - \lambda_2 e^{\lambda_2 T} \right] \right\} \mu \\
&+(1+G_2) e^{(\lambda_1 + \lambda_2)T} = 0.
\end{aligned} \tag{11.54}$$

System (11.49) has a non-trivial T-periodic solution if and only if at least one multiplier is equal to 1, then substitute μ in (11.54) by 1, we have

$$1 + G_2 = \frac{\lambda_1 - \lambda_2 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T}}{\lambda_1 e^{\lambda_1 T} - \lambda_2 e^{\lambda_2 T} - (\lambda_1 - \lambda_2) e^{(\lambda_1 + \lambda_2)T}}. \tag{11.55}$$

Then we have the following two cases.

1. *(11.55) holds*. In this case $\text{rank}(M - I) = 1$ and the system (11.50) has one linearly independent T-periodic solution. It follows from the first condition of Theorem 9.1.6 we know that the following adjoint system of (11.50)

$$\begin{cases} \dot{\boldsymbol{x}} = -A^*\boldsymbol{x}, & t = \tau_k, \\ \Delta\boldsymbol{x} = -(I + B^*)^{-1}B^*\boldsymbol{x}, & t = \tau_k, \end{cases} \tag{11.56}$$

also has one linear independent T-periodic solution $\boldsymbol{x}_T(t)$ whose initial condition $\boldsymbol{x}_T(0^+) \triangleq (x_1(0), x_2(0))^\top$ is given by

$$\begin{aligned} \boldsymbol{x}_T(0^+) &= (I + B^*)^{-1}e^{-A^*T}\boldsymbol{x}_T(0^+) \\ &= \frac{1}{\nu_1 - \nu_2} \\ &\quad \times \begin{pmatrix} \nu_1 e^{\nu_2 T} - \nu_2 e^{\nu_1 T} & e^{\nu_1 T} - e^{\nu_2 T} \\ -\frac{\nu_1\nu_2(e^{\nu_1 T} - e^{\nu_2 T})}{1+G_2} & \frac{\nu_1 e^{\nu_1 T} - \nu_2 e^{\nu_2 T}}{1+G_2} \end{pmatrix} \boldsymbol{x}_T(0^+), \end{aligned} \tag{11.57}$$

where

$$\nu_{1,2} = \frac{G_3 \pm \sqrt{G_3^2 - 4G_1}}{2} \tag{11.58}$$

are two eigenvalues of $-A^*$.
Assume that (11.57) has the following nonzero solution:

$$\boldsymbol{x}_T(0^+) \triangleq (x_1(0), x_2(0))^\top \tag{11.59}$$

then we can construct a non-trivial T-periodic solution of (11.56) as

$$\begin{aligned} \boldsymbol{x}_T(t) &\triangleq \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &= e^{-A^*t}\boldsymbol{x}_T(0^+) \\ &= \frac{1}{\nu_1 - \nu_2} \\ &\quad \times \begin{pmatrix} x_1(0)(\nu_1 e^{\nu_2 t} - \nu_2 e^{\nu_1 t}) + x_2(0)(e^{\nu_1 t} - e^{\nu_2 t}) \\ -x_1(0)(\nu_1\nu_2(e^{\nu_1 t} - e^{\nu_2 t})) + x_2(0)(\nu_1 e^{\nu_1 t} - \nu_2 e^{\nu_2 t}) \end{pmatrix}, \\ &\quad t \in (0, T]. \end{aligned} \tag{11.60}$$

Therefore, it follows from condition 2 of Theorem 9.1.6 that system (11.49) has a T-periodic solution if and only if c_k and $i(t)$ satisfy (9.68); namely,

$$\int_0^T x_2(t)i(t)dt + \sum_{0 \le \tau_k < T} x_2(\tau_k^+)c_k = 0.$$

2. *(11.55) does not hold.* In this case, from Theorem 9.1.5 we know that (11.49) has a unique T-periodic solution. Furthermore, in view of (11.54), if the condition

$$\mu_1^2 + \mu_2^2 = (\mu_1 + \mu_2)^2 - 2\mu_1\mu_2 < 1; \tag{11.61}$$

namely,

$$\left[\frac{1}{\lambda_1 - \lambda_2}\left\{\lambda_1 e^{\lambda_2 T} - \lambda_2 e^{\lambda_1 T} + (1 + G_2)\left[\lambda_1 e^{\lambda_1 T} - \lambda_2 e^{\lambda_2 T}\right]\right\}\right]^2 \\ -2(1 + G_2)e^{-(\lambda_1+\lambda_2)T} < 1 \tag{11.62}$$

is satisfied, then multipliers of (11.50) satisfy $|\mu_{1,2}| < 1$. Thus, system (11.49) is asymptotically stable.

11.6 T-Periodic Solutions of First-order Nanoelectronic Circuits

11.6.1 Circuit Consisting of Linear IVCCS

Let us study the existence of T-periodic solutions of the first-order SETJ circuit shown in Fig. 11.6. In this circuit, there is an impulsive voltage control current source (IVCCS) connected in series with the a linear capacitor C.

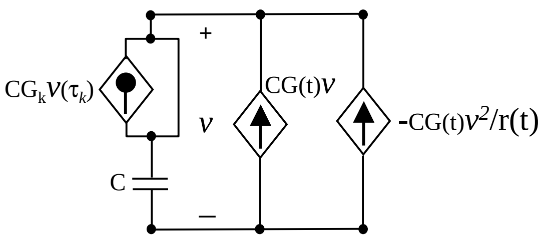


Fig. 11.6. The circuit block diagram of a first-order nanoelectronic circuit with impulsive voltage control current source controlled by a clock signal $\tau_k, k \in \mathbb{N}$.

Let $\tau_k, k \in \mathbb{N}$ be the clock signal that control the IVCCS, when $t \neq \tau_k$ we have

$$C\frac{dv}{dt} = CG(t)v - CG(t)v^2/r(t), \tag{11.63}$$

where $r \in \mathcal{PC}[\mathbb{R}, \mathbb{R}]$,

$$\inf_{t\in[0,T]} r(t) > 0,$$

and $G \in \mathcal{PC}[\mathbb{R}, \mathbb{R}]$ and $G(t) > 0$ for $t \geq 0$.

Let $q(t)$ be charge in C. When $t = \tau_k$ we have

$$\begin{aligned}\Delta q &= C\Delta v\\ &= \int_{-\infty}^{\infty} CG_k v(\tau_k)\delta(\tau_k)dt = CG_k v(\tau_k), \qquad \Rightarrow\\ \Delta v &= G_k v(\tau_k).\end{aligned} \tag{11.64}$$

Let us assume that $1 + G_k > 0$, $k \in \mathbb{N}$.

The state equation of this circuit is given by

$$\begin{aligned}\dot{v} &= G(t)v(t) - G(t)v^2(t)/r(t), \quad t \neq \tau_k,\\ \Delta v &= G_k v(\tau_k), \quad t = \tau_k, \quad k \in \mathbb{N}.\end{aligned} \tag{11.65}$$

Let us assume that there are $T > 0$ and $\varrho \in \mathbb{N}$ such that for $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$

$$G(t+T) = G(t),\ r(t+T) = r(t),\ \tau_{k+\varrho} = \tau_k + T,\ G_{k+\varrho} = G_k. \tag{11.66}$$

For the purpose of logic operations, we may need to find conditions under which the T-periodic solution of circuit (11.65) either positive(logic 1) or negative(logic 0). Under this condition we assume that $v(t) \neq 0$ for all $t \in \mathbb{R}_+$. Let $x(t) = 1/v(t)$ then from (11.65) it follows that

$$\begin{aligned}\dot{x} &= -G(t)x(t) + G(t)/r(t), \quad t \neq \tau_k,\\ x(\tau_k^+) &= \frac{1}{1+G_k}x(\tau_k), \quad t = \tau_k, \quad k \in \mathbb{N}.\end{aligned} \tag{11.67}$$

The reference system is given by

$$\begin{aligned}\dot{x} &= -G(t)x(t), \quad t \neq \tau_k,\\ x(\tau_k^+) &= \frac{1}{1+G_k}x(\tau_k), \quad t = \tau_k, \quad k \in \mathbb{N}\end{aligned} \tag{11.68}$$

whose Cauchy matrix is given by

$$\Psi(t,s) = \exp\left(-\int_s^t G(\tau)d\tau\right) \prod_{s \leq t < t} \frac{1}{1+G_k}. \tag{11.69}$$

Then we have the following solution of system (11.67):

$$x(t) = \Psi(t,0)x(0) + \int_0^t \Psi(t,s)G(s)/r(s)ds \tag{11.70}$$

which is a T-periodic solution if

$$[1 - \Psi(t,0)]x(0) = \int_0^T \Psi(T,s)G(s)/r(s)ds. \tag{11.71}$$

Let us assume that the multiplier μ of the reference system satisfy

$$\mu = \Psi(T,0) = \exp\left(-\int_0^T G(\tau)d\tau\right)\prod_{k=1}^{\varrho}\frac{1}{1+G_k} < 1 \tag{11.72}$$

and

$$\int_0^T \Psi(T,s)G(s)/r(s)ds > 0 \tag{11.73}$$

then (11.71) has a unique solution $x(0) = x_0 > 0$. Furthermore, from (11.70) and the definition of $\Psi(t,s)$ we know that $x(t,0,x_0)$ is positive. Therefore $v_T(t) = 1/x(t,0,x_0)$ is a positive T-periodic solution of circuit (11.65).

11.6.2 Circuit Consisting of Nonlinear IVCCS

Let us study the existence of T-periodic solutions of the first-order SETJ circuit shown in Fig. 11.7. In this circuit, there is a nonlinear impulsive voltage control current source (IVCCS) connected in series with the a linear capacitor C. Also observe that if $G_0 > 0$ then there is a negative resistor in this circuit.

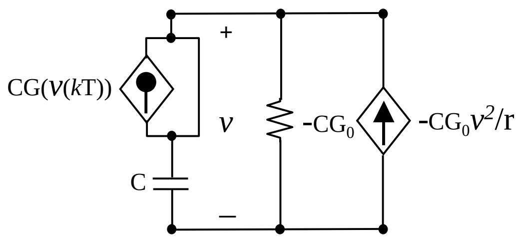


Fig. 11.7. The circuit block diagram of a first-order nanoelectronic circuit with impulsive voltage control current source controlled by a clock signal $kT, k \in \mathbb{N}$.

Let $kT, k \in \mathbb{N}$ be the clock signal that control the IVCCS, when $t \neq kT$ we have

$$C\frac{dv}{dt} = CG_0v - CG_0v^2/r. \tag{11.74}$$

Let us assume that $r > 0$ and $G_0 > 0$. Let $q(t)$ be charge in C. When $t = kT$ we have

$$\begin{aligned}\Delta q &= C\Delta v\\ &= \int_{-\infty}^{\infty} CG(v(kT))\delta(kT)dt = CG(v(kT)), \qquad \Rightarrow\\ \Delta v &= G(v(kT))\end{aligned} \tag{11.75}$$

where $G \in C^1[\mathbb{R}_+, \mathbb{R}]$ and $G(v(kT)) > -v(kT)$ for $v(kT) \in \mathbb{R}_+$. The state equation of this circuit is given by

$$\begin{aligned} \dot{v} &= G_0 v - G_0 v^2/r, , \quad t \neq kT, \\ \Delta v &= G(v(kT)), \quad t = kT, \quad k \in \mathbb{N}. \end{aligned} \tag{11.76}$$

Let us assume that there are $T > 0$ and $\varrho \in \mathbb{N}$ such that for $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$

$$G(t+T) = G(t),\ r(t+T) = r(t),\ \tau_{k+\varrho} = \tau_k + T,\ G_{k+\varrho} = G_k. \tag{11.77}$$

If we use the nanoelectronic circuit (11.76) to represent the state of logic 1, then we need to find a positive T-periodic solution $v_T(t) = v_T(t, 0, v_0)$ of (11.76) with $v_T(0) = v_0 > 0$. When $t \in (0, T]$ we have

$$v_T(t) = \frac{v_0}{e^{-G_0 t} + (1 - e^{-G_0 t}) v_0 / r}. \tag{11.78}$$

Since $v_T(t)$ is a T-periodic solution, we have the following conditions:

$$v_T(T) = \frac{v_0}{e^{-G_0 T} + (1 - e^{-G_0 T}) v_0 / r}, \quad v_0 = v_T(T) + G(v_T(T)) \tag{11.79}$$

from which we can find a positive $v_T(T)$ by

$$v_T(T) = \frac{r v_T(T) + r G(v_T(T))}{r e^{-G_0 T} + (1 - e^{-G_0 T})[v_T(T) + G(v_T(T))]}. \tag{11.80}$$

Then we have the following variational equation with respect to $v_T(t)$ and circuit (11.76):

$$\begin{aligned} \dot{x} &= G_0 x - 2 G_0 v_T(t) x / r, , \quad t \neq kT, \\ \Delta x &= \frac{\partial G(v_T(kT))}{\partial x}, \quad t = kT, \quad k \in \mathbb{N} \end{aligned} \tag{11.81}$$

whose multiplier is given by

$$\mu = \exp\left\{ \int_0^T (G_0 - 2 G_0 v_T(t)/r)\, dt \right\} \left(1 + \frac{\partial G(v_T(kT))}{\partial x} \right). \tag{11.82}$$

Then the existence of a positive T-periodic solution is given by $0 < \mu < 1$.

11.7 Nanoelectronic Circuit Consisting of Nonlinear IVCCS

Let us study the existence of T-periodic solutions of the second-order SETJ circuit shown in Fig. 11.8. In this circuit, there is a nonlinear impulsive voltage

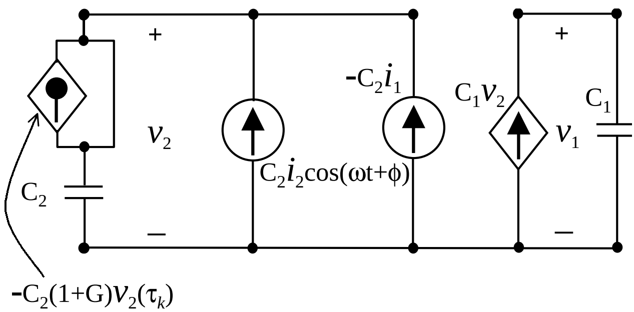


Fig. 11.8. The circuit block diagram of a second-order nanoelectronic circuit with impulsive voltage control current source.

control current source (IVCCS) connected in series with the a linear capacitor C_2.

The IVCCS is controlled by impulse moments $\{\tau_k\}$ satisfying

$$v_1(\tau_k) = 0, \quad k \in \mathbb{N}.$$

When $t \neq \tau_k$ we have

$$\begin{aligned} C_1 \frac{dv_1}{dt} &= C_1 v_2(t), \\ C_2 \frac{dv_2}{dt} &= -C_2 i_1 + C_2 i_2 \cos(\omega t + \phi). \end{aligned} \tag{11.83}$$

Let us assume that $i_1 > 0$, $i_2 > 0$, $\omega > 0$ and $\phi \in \mathbb{R}$ is a constant phase. Let $q_1(t)$ and $q_2(t)$ be charges in C_1 and C_2, respectively. When $t = \tau_k$; namely, $v_1(t) = 0$, we have

$$\begin{aligned} \Delta q_1 &= C_1 \Delta v_1 = 0, \\ \Delta q_2 &= C_2 \Delta v_2 \\ &= \int_{-\infty}^{\infty} -C_2(1+G) v_2(\tau_k)\delta(\tau_k)dt = -C_2(1+G)v_2(\tau_k), \qquad \Rightarrow \\ \Delta v_1 &= 0, \quad \Delta v_2 = -(1+G)v_2. \end{aligned} \tag{11.84}$$

Let us assume that $G \in (0, 1)$. The state equation of this circuit is given by

$$\begin{aligned} &\begin{cases} \dot{v}_1 = v_2 \\ \dot{v}_2 = -i_1 + i_2 \cos(\omega t + \phi) \end{cases}, v_1(t) \neq 0, \\ &\begin{cases} \Delta v_1 = 0 \\ \Delta v_2 = -(1+G)v_2 \end{cases}, v_1(t) = 0. \end{aligned} \tag{11.85}$$

To represent logic 1, let us find a nonnegative T-periodic solution, $\boldsymbol{v}_T(t) = (v_1^T(t), v_2^T(t))^\top$, of circuit (11.85). Let us assume that there is only one impulsive in one period and we have the following initial condition

$$v_1(0^+) = 0, \quad v_2(0^+) = v_{20} > 0, \tag{11.86}$$

then for $t \in (0, T]$ we have

$$\begin{aligned} v_1^T(t) &= -i_1 t^2/2 + \left(v_{20} - \frac{i_2}{\omega}\sin(\phi)\right)t + \frac{i_2}{\omega^2}\cos(\phi) - \frac{i_2}{\omega^2}\cos(\omega t + \phi), \\ v_2^T(t) &= -i_1 t + v_{20} - \frac{i_2}{\omega}\sin(\phi) + \frac{i_2}{\omega}\sin(\omega t + \phi). \end{aligned} \tag{11.87}$$

Since the AC current source is of period $2\pi/\omega$, then T will have the values of

$$T = m\frac{2\pi}{\omega}, \quad m \in \mathbb{N}. \tag{11.88}$$

Then from (11.86) and (11.87) we have the following conditions for $\boldsymbol{v}_T(T) = \boldsymbol{v}_T(0^+)$:

$$\begin{aligned} i_1 T^2/2 &= \left(v_{20} - \frac{i_2}{\omega}\sin(\phi)\right)T, \\ v_2^T(T) &= -i_1 T + v_{20}. \end{aligned} \tag{11.89}$$

From the behavior of the IVCCS we know that

$$v_{20} = v_2^T(T^+) = [1 - (1+G)]v_2^T(T) = -Gv_2^T(T). \tag{11.90}$$

Thus from (11.89) and (11.90) we have the following conditions:

$$v_2^T(T) = -\frac{i_1 T}{1+G}, \tag{11.91}$$

$$v_{20} = \frac{i_1 TG}{1+G}, \tag{11.92}$$

$$i_2\sin(\phi) = -i_1\frac{\omega T(1-G)}{2(1+G)} \triangleq -i_1 R. \tag{11.93}$$

To study the stability of $\boldsymbol{v}_T$, we use the following variation system of circuit (11.85) with respect to $\boldsymbol{v}_T$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A}\boldsymbol{x}, \qquad t \neq mT, \\ \Delta\boldsymbol{x} &= \underbrace{\begin{pmatrix} -1-G & 0 \\ (1+G)(-i_1 + i_2\cos(\phi))/v_2^T(T) & -1-G \end{pmatrix}}_{B}\boldsymbol{x}, \\ & t = mT, \quad m \in \mathbb{N}. \end{aligned} \tag{11.94}$$

The monodromy matrix, M, is given by

$$M = (I + B)e^{AT} = \begin{pmatrix} -G & -GT \\ p & pT - G \end{pmatrix} \tag{11.95}$$

where

$$p = \frac{(1+G)^2(i_1 - i_2\cos(\phi))}{i_1 T}. \tag{11.96}$$

Then the multipliers, $\mu_{1,2}$, are given by

$$\mu^2 + (2G - pT)\mu + G^2 = 0. \tag{11.97}$$

$\boldsymbol{v}_T(t)$ is asymptotically stable if and only if $|\mu_{1,2}| < 1$. If $|\mu_{1,2}| < 1$, then from (11.97) we have

$$\begin{aligned} 0 &= \mu^2 + (2G - pT)\mu + G^2 \\ &\le \mu^2 + |(2G - pT)\mu| + G^2 \\ &< 1 + |2G - pT| + G^2 \end{aligned} \tag{11.98}$$

from which we have

$$0 < i_2\cos(\phi) < i_1\left[1 + \left(\frac{1-G}{1+G}\right)^2\right]. \tag{11.99}$$

In view (11.93) and (11.99) we have

$$|\mu_{1,2}| < 1 \quad \Longleftrightarrow \quad \frac{i_2}{\sqrt{R^2 + \left[1 + \left(\frac{1-G}{1+G}\right)^2\right]^2}} < i_1 < i_2/R \tag{11.100}$$

which gives the conditions for the asymptotic stability of the T-periodic solution.

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